A CHAPTER

IN THE

INTEGRAL CALCULUS.

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PREFACE.

The present pamphlet is intended to be used by mathematical students as supplementary to an ordinary treatise on the Integral Calculus.

A method of Integration is proposed which makes this operation, always of a tentative nature, more systematic and certain; and in this method the hyperbolic functions are used freely in conjunction with the circular trigonometrical functions.

Considering their importance in Applied Mathematics, the hyperbolic functions have not received adequate treatment in ordinary text-books; to illustrate this importance, a digression has been made on their principal properties, illustrated by examples of their application.

The recent Cambridge examination papers have been consulted for examples, to exhibit the methods of integration explained in this pamphlet, which, it is hoped, will prove useful and interesting to the mathematical student.

CONTENTS.

A CHAPTER IN THE INTEGRAL CALCULUS.

Introduction.

PRESUMING that the reader is already familiar with the general methods of Integration, as given in text-books on the Integral Calculus, it is proposed in the following pages to explain how the process of Integration may in some cases be made more systematic, so that the result of the integration may be more readily written down, when the function to be integrated is once presented for that purpose.

The process of Integration is necessarily of a tentative nature, depending on a previous knowledge of Differentiation; and in general the most convenient order of the mental operations required for the integration of a given function will be found to be:

(i.) To guess the function required for the integration;
(ii.) To assign the argument of this function;
(iii.) To write down the proper constant or numerical factors of the integrated function.

Of these three operations, the first is of the most fundamental importance, as depending on the principles of the Calculus, but it is the second operation which presents the greatest practical difficulty, while the third only requires verification by a mental differentiation.

To lessen the difficulty of the second operation it is the object of these pages to show that it is convenient to take, either the function to be integrated, or *constituents* of this function, as the argument of the integrated function; for instance, when $\int \sec x\, dx$ is required, to express the result as a function of $\sec x$, and when

$$\int \frac{dx}{(x-p)\sqrt{(ax^2+2bx+c)}}, \text{ or } \int \frac{Mx+N}{(Ax^2+2Bx+C)\sqrt{(ax^2+2bx+c)}}\, dx,$$

is required, to express the result as a function of

$$y = \frac{\sqrt{(ax^2+2bx+c)}}{x-p}, \text{ or } \sqrt{\left(\frac{ax^2+2bx+c}{Ax^2+2Bx+C}\right)},$$

by changing the independent variable from x to y by these substitutions in the ordinary manner; and generally, to express $\int \phi(x)\, dx$ as a function of $\phi(x)$, or of the constituents of $\phi(x)$.

General Integration of Rational and Irrational Algebraical Functions.

1. Presuming, then, that the reader has already studied the Integral Calculus to a certain extent, let us begin by considering the most general algebraical function of a variable x, of which the integral with respect to x (excluding *pseudo-elliptic* integrals) can be expressed by algebraical, circular, exponential and logarithmic, or hyperbolic functions of x.

The most general function is of the form

$$\frac{A+B\sqrt{R}}{P+Q\sqrt{R}},$$

when A, B, P, Q are rational integral functions of x, and R is a linear or quadratic function of x, and therefore of the form

$$ax^2+2bx+c.$$

If the degree of R is higher than the second, and the integration can still be expressed by algebraical, circular, or hyperbolic functions, the integral is called *pseudo-elliptic*, and it can always by a suitable substitution be reduced to an integration in which the degree of R, in the irrational part $\sqrt{R}$, is of the second degree at most.

2. The first operation is to rationalise the denominator; then

$$\frac{A+B\sqrt{R}}{P+Q\sqrt{R}}=\frac{(A+B\sqrt{R})(P-Q\sqrt{R})}{P^2-Q^2R}=\frac{AP-BQR}{P^2-Q^2R}+\frac{BP-AQ}{P^2-Q^2R}\sqrt{R},$$

of the form
$$\frac{M}{D}+\frac{N}{D}\,\frac{1}{\sqrt{R}},$$

where
$$M=AP-BQR,$$
$$N=BPR-AQR,$$
$$D=P^2-Q^2R,$$

rational integral functions of x.

Then
$$\int\frac{A+B\sqrt{R}}{P+Q\sqrt{R}}\,dx=\int\frac{M}{D}\,dx+\int\frac{N}{D}\,\frac{dx}{\sqrt{R}}.$$

3. To integrate the rational fraction M/D, this function is split up into its partial fractions, in the manner explained in treatises on Algebra, the quotient, if any, being first obtained by division, and then the integration of each term is in general easily effected.

4. Next, the integral $\int \frac{N}{D}\frac{dx}{\sqrt{R}}$ can be immediately made to depend upon the integration of another rational fraction, by means of a simple substitution.

Three cases must be considered, according to the form of

$$R \equiv ax^2+2bx+c.$$

(i.) If $ac-b^2$ is positive, and therefore also a and c are positive, in order that R should be positive, and $\sqrt{R}$ real, put

$$\frac{\sqrt{a}\sqrt{R}}{\sqrt{(ac-b^2)}}=\frac{1+y^2}{1-y^2},$$

then
$$\frac{ax+b}{\sqrt{(ac-b^2)}}=\frac{2y}{1-y^2},$$

and
$$\frac{a\,dx}{\sqrt{(ac-b^2)}}=2\frac{1+y^2}{(1-y^2)^2}dy,$$

so that
$$\frac{\sqrt{a}\,dx}{\sqrt{R}}=\frac{2\,dy}{1-y^2}.$$

Then the integral $\int \frac{N}{D}\frac{dx}{\sqrt{R}}=\int \frac{N'}{D'}dy,$

where N' and D' are certain rational integral functions of y, and the integration is effected as before, by resolving N'/D' into its partial fractions.

(ii.) If $ac-b^2$ is negative but a is positive, then, putting

$$\frac{\sqrt{a}\sqrt{R}}{\sqrt{(b^2-ac)}}=\frac{2y}{1-y^2},$$

$$\frac{ax+b}{\sqrt{(b^2-ac)}}=\frac{1+y^2}{1-y^2},$$

and
$$\frac{\sqrt{a}\,dx}{\sqrt{R}}=\frac{2\,dy}{1-y^2};$$

so that, again
$$\int \frac{N}{D}\frac{dx}{\sqrt{R}}=\int \frac{N'}{D'}dy,$$

where N' and D' are rational integral functions of y.

(iii.) If $ac-b^2$ is negative, and a is negative, put

$$\frac{\sqrt{(-a)}\sqrt{R}}{\sqrt{(b^2-ac)}}=\frac{2y}{1+y^2};$$

then
$$\frac{ax+b}{\sqrt{(b^2-ac)}}=\frac{1-y^2}{1+y^2},$$

$$\frac{-a\,dx}{\sqrt{(b^2-ac)}}=\frac{4y\,dy}{(1+y^2)^2},$$

and
$$\frac{\sqrt{(-a)}\,dx}{\sqrt{R}}=\frac{2dy}{1+y^2},$$

so that again the integral is rationalised; and in each case the substitution employed involves the constituent function R.

Subsequently, however, we shall find it preferable not to use these substitutions, but to reduce the integral $\int \frac{N}{D}\frac{dx}{\sqrt{R}}$ by resolving the rational function N/D into its partial fractions, and considering the integral of each term.

The Resolution of a Rational Function into its Partial Fractions.

5. Let us make a digression on this question of Algebra, so far as it is required in the Integral Calculus.

Denoting by N the numerator and D the denominator of a rational fraction, so that N and D are *rational integral* functions of x, that is, each is of the form $Ax^n+Bx^{n-1}+Cx^{n-2}+\ldots+Px+Q$,

where n is an integer; then, if the degree of N is greater than the degree of D, the quotient must first be obtained by dividing N by D. After this has been done we may denote the remainder by N, so that henceforth the degree of N may be supposed less than that of D.

Next, suppose D to be split up into its real linear and quadratic factors, single or repeated; then

(i.) Corresponding to a single linear factor $x-a$, suppose, we must assume a corresponding single partial fraction $\frac{A}{x-a}$; and then

$$A=\frac{N(x-a)}{D}\text{ for the value } x=a;$$

the corresponding integral being $A\log(x-a)$.

(ii.) Corresponding to a repeated linear factor $(x-b)^m$, we must assume m partial fractions

$$\frac{B_m}{(x-b)^m}+\frac{B_{m-1}}{(x-b)^{m-1}}+\ldots+\frac{B_1}{x-b};$$

and then, to determine the B's, put $x-b=y$, and expand $N(x-b)^m/D$ in ascending powers of y as far as y^{m-1}; the successive coefficients will be $B_m, B_{m-1}, \ldots B_1$.

The integral corresponding to $\frac{B_r}{(x-b)^r}$ is $-\frac{B_r}{(r-1)(x-b)^{r-1}}$, and to $\frac{B_1}{x-b}$ is $B_1\log(x-b)$, as above.

(iii.) Corresponding to a quadratic factor of the form $(x-a)^2+\beta^2$, splitting up into imaginary linear factor $x-a\pm i\beta$, we must assume a corresponding partial fraction

$$\frac{Lx+M}{(x-a)^2+\beta^2},$$

which may be supposed to result from the coalescence of two conjugate imaginary partial fractions

$$\frac{A+iB}{x-a+i\beta}+\frac{A-iB}{x-a-i\beta}=2\frac{A(x-a)+B\beta}{(x-a)^2+\beta^2};$$

the integral of which is

$$A\log\{(x-a)^2+\beta^2\}+2B\tan^{-1}\frac{x-a}{\beta}.$$

(iv.) Corresponding to a quadratic factor, repeated m times, $\{(x-a)^2+\beta^2\}^m$, it is customary to determine m partial fractions of the form

$$\frac{L_r x+M_r}{\{(x-a)^2+\beta^2\}^r},$$

but this method is not of practical use, and a better plan is to determine the conjugate imaginary partial fractions of the form

$$\frac{B+iC}{(x-a+i\beta)^r}+\frac{B-iC}{(x-a-i\beta)^r},$$

which integrated give

$$-\frac{1}{r-1}\left\{\frac{B+iC}{(x-a+i\beta)^{r-1}}+\frac{B-iC}{(x-a-i\beta)^{r-1}}\right\}$$

$$=-\frac{1}{r-1}\frac{\left[\begin{array}{l}B\{(x-a+i\beta)^{r-1}+(x-a-i\beta)^{r-1}\}\\ \qquad -iC\{(x-a+i\beta)^{r-1}-(x-a-i\beta)^{r-1}\}\end{array}\right]}{\{(x-a)^2+\beta^2\}^r}$$

easily reducible to a real form.

The Integration of the Irrational Function.

6. Now, resuming the consideration of the integral

$$\int\frac{N}{D}\frac{dx}{\sqrt{R}},$$

and supposing the rational function N/D resolved into partial fractions, consider first the integral corresponding to a single partial fraction of the form $\dfrac{A}{x-p}$, $x-p$ being a single linear factor of D.

The corresponding integral

$$\int \frac{dx}{(x-p)\sqrt{R}}$$

is usually reduced by means of the substitution $x-p=\frac{1}{y}$, and then becomes of the form $\int dy/\sqrt{R'}$, where R' is of the form $a'y^2+2b'y+c'$.

But we shall find it better, in development of the general idea of these pages, to employ the substitution

$$y=\frac{\sqrt{R}}{x-p},$$

the quotient of the constituents $\sqrt{R}$ and $x-p$; then

$$(ap^2+2bp+c)\,y^2-(ac-b^2)=\left\{\frac{(ap+b)\,x+bp+c}{x-p}\right\}^2,$$

and, taking the minus sign with the radical

$$\frac{(ap+b)\,x+bp+c}{x-p}=-\sqrt{\{(ap^2+2bp+c)\,y^2-ac+b^2\}},$$

$$\frac{dx}{(x-p)^2}=\frac{y\,dy}{\sqrt{\{(ap^2+2bp+c)\,y^2-ac+b^2\}}},$$

so that $$\int\frac{dx}{(x-p)\sqrt{R}}=\int\frac{dy}{\sqrt{\{(ap^2+2bp+c)\,y^2-ac+b^2\}}}.$$

7. The form of the integral will be different according to the signs of $ap^2+2bp+c$ and $ac-b^2$.

The first cannot be negative and the second positive, because then R would be negative and $\sqrt{R}$ consequently imaginary.

Suppose $ap^2+2bp+c$ negative and $ac-b^2$ negative, then

$$\int\frac{dx}{(x-p)\sqrt{R}}=\frac{1}{\sqrt{(-ap^2-2bp-c)}}\sin^{-1}\frac{y\sqrt{(-ap^2-2bp-c)}}{\sqrt{(b^2-ac)}}$$

$$=\frac{1}{\sqrt{(-ap^2-2bp-c)}}\sin^{-1}\frac{\sqrt{(ax^2+2bx+c)}\sqrt{(-ap^2-2bp-c)}}{(x-p)\sqrt{(b^2-ac)}},$$

the integral being thus expressed as a function of the quotient of the constituents $\sqrt{R}$ and $x-p$.

8. When $ap^2+2bp+c$ is positive, then

$$\int\frac{dx}{(x-p)\sqrt{R}}=\frac{1}{\sqrt{(ap^2+2bp+c)}}\int\frac{dy}{\sqrt{(y^2+\lambda)}},$$

where λ is some constant $=-(ac-b^2)/(ap^2+2bp+c)$, which may be positive or negative.

When only the logarithmic function is employed, then, as is well known,

$$\int \frac{dy}{\sqrt{(y^2+\lambda)}} = \log\left\{\sqrt{(y^2+\lambda)}+y\right\},$$

requiring no distinction between positive and negative values of λ;

thus
$$\int \frac{dy}{\sqrt{(y^2+m^2)}} = \log\left\{\sqrt{(y^2+m^2)}+y\right\},$$

$$\int \frac{dy}{\sqrt{(y^2-m^2)}} = \log\left\{y+\sqrt{(y^2-m^2)}\right\};$$

but, comparing these with

$$\int \frac{dy}{\sqrt{(m^2-y^2)}} = \sin^{-1}\frac{y}{m} \text{ or } -\cos^{-1}\frac{y}{m},$$

we notice a great complication in the argument of the logarithmic function, compared with the argument of the inverse circular functions.

The Hyperbolic Functions.

9. This complication is avoided, and great symmetry is obtained by the introduction of the *hyperbolic functions*, direct and inverse.

The definitions of these functions are

$$\cosh u = \tfrac{1}{2}(e^u+e^{-u}), \quad \sinh u = \tfrac{1}{2}(e^u-e^{-u}),$$

$\cosh u$ being called the *hyperbolic cosine*, and $\sinh u$ the *hyperbolic sine* of u.

By analogy with the other circular functions, $\dfrac{\sinh u}{\cosh u} = \dfrac{e^u-e^{-u}}{e^u+e^u}$ is called the *hyperbolic tangent* of u, and is denoted by $\tanh u$, and so on.

Then
$$\frac{d\cosh u}{du} = \sinh u, \quad \frac{d\sinh u}{du} = \cosh u,$$

also
$$\cosh^2 u - \sinh^2 u = 1;$$

and therefore
$$\frac{d\cosh u}{du} = \sqrt{(\cosh^2 u - 1)},$$

$$\frac{d\sinh u}{du} = \sqrt{(\sinh^2 u + 1)}.$$

10. Putting $\cosh u = x$, then u is the inverse hyperbolic cosine of x, which we shall denote by the notation

$$u = \cosh^{-1} x,$$

a notation now common in America (W. E. Byerly, *Integral Calculus*);

it will be found employed by Dr. Ferrers, in the *Quarterly Journal of Mathematics*, Vol. XVIII., "Distribution of Electricity on a Bowl"; and other writers on applied mathematics.

Bertrand (*Calcul Intégral*, p. 15) proposes the notation sect. cos hyp. x for $\cosh^{-1} x$, by analogy with the continental $\operatorname{arc.cos} x$ for $\cos^{-1} x$; but the lengthiness of this notation has apparently prevented him from using it subsequently in his treatise.

Hence
$$\frac{dx}{du} = \sqrt{(x^2-1)},$$

or
$$\frac{d\cosh^{-1} x}{dx} = \frac{1}{\sqrt{(x^2-1)}},$$

so that
$$\int_1^x \frac{dx}{\sqrt{(x^2-1)}} = \cosh^{-1} x;$$

and similarly
$$\frac{d\sinh^{-1} x}{dx} = \frac{1}{\sqrt{(x^2+1)}}$$

and
$$\int_0^x \frac{dx}{\sqrt{(x^2+1)}} = \sinh^{-1} x.$$

Expressed by the hyperbolic logarithmic function,
$$\cosh^{-1} x = \log\{x+\sqrt{(x^2-1)}\},$$
$$\sinh^{-1} x = \log\{\sqrt{(x^2+1)}+x\},$$

so that now
$$\int \frac{dy}{\sqrt{(y^2+m^2)}} = \sinh^{-1}\frac{y}{m} = \cosh^{-1}\frac{\sqrt{(y^2+m^2)}}{m},$$
$$\int \frac{dy}{\sqrt{(y^2-m^2)}} = \cosh^{-1}\frac{y}{m} = \sinh^{-1}\frac{\sqrt{(y^2-m^2)}}{m}.$$

Now, expressed as a function of the argument,
$$y = \frac{\sqrt{R}}{x-p},$$

where $R = ax^2+2bx+c$, and $ap^2+2bp+c$ is positive,
$$\int \frac{dx}{(x-p)\sqrt{R}} = \frac{1}{\sqrt{(ap^2+2bp+c)}}\cosh^{-1}\frac{\sqrt{R}\sqrt{(ap^2+2bp+c)}}{(x-p)\sqrt{(ac-b^2)}},$$

or
$$= \frac{1}{\sqrt{(ap^2+2bp+c)}}\sinh^{-1}\frac{\sqrt{R}\sqrt{(ap^2+2bp+c)}}{(x-p)\sqrt{(b^2-ac)}},$$

according as $ac-b^2$ is positive or negative.

11. With this hyperbolic function notation, $a > \beta$,

$$\int_a^x \frac{dx}{\sqrt{(x-a.x-\beta)}} = 2\sinh^{-1}\sqrt{\frac{x-a}{a-\beta}} = 2\cosh^{-1}\sqrt{\frac{x-\beta}{a-\beta}}$$

$$= 2\tanh^{-1}\sqrt{\frac{x-a}{x-\beta}} = 2\log\frac{\sqrt{(x-a)}+\sqrt{(x-\beta)}}{\sqrt{(a-\beta)}}$$

$$= \sinh^{-1}\frac{2\sqrt{(x-a.x-\beta)}}{a-\beta};$$

$$\int_x^\beta \frac{dx}{\sqrt{(x-a.x-\beta)}} = 2\cosh^{-1}\sqrt{\frac{a-x}{a-\beta}} = 2\sinh^{-1}\sqrt{\frac{\beta-x}{a-\beta}}$$

$$= 2\tanh^{-1}\sqrt{\frac{\beta-x}{a-x}} = 2\log\frac{\sqrt{(a-x)}+\sqrt{(\beta-x)}}{\sqrt{(a-\beta)}}$$

$$= \sinh^{-1}\frac{2\sqrt{(a-x.\beta-x)}}{a-\beta};$$

$$\int_a^x \frac{dx}{\sqrt{\{(x-a)^2+\beta^2\}}} = \cosh^{-1}\frac{\sqrt{\{(x-a)^2+\beta^2\}}}{\beta};$$

while, for the corresponding circular integrals,

$$\int_\beta^x \frac{dx}{\sqrt{(a-x.x-\beta)}} = 2\sin^{-1}\sqrt{\frac{x-\beta}{a-\beta}} = 2\cos^{-1}\sqrt{\frac{a-x}{a-\beta}}$$

$$= 2\tan^{-1}\sqrt{\frac{x-\beta}{a-x}} = \sin^{-1}\frac{2\sqrt{(a-x.x-\beta)}}{a-\beta};$$

$$\int_x^a \frac{dx}{\sqrt{(a-x.x-\beta)}} = 2\cos^{-1}\sqrt{\frac{x-\beta}{a-\beta}} = 2\sin^{-1}\sqrt{\frac{a-x}{a-\beta}}$$

$$= 2\tan^{-1}\sqrt{\frac{a-x}{x-\beta}} = \sin^{-1}\frac{2\sqrt{(a-x.x-\beta)}}{a-\beta}.$$

12. Generally, denoting $ax^2+2bx+c$ by R,

$$\int\frac{dx}{\sqrt{R}} = \frac{1}{\sqrt{a}}\cosh^{-1}\frac{\sqrt{a}\sqrt{R}}{\sqrt{(ac-b^2)}},$$

or

$$= \frac{1}{\sqrt{a}}\sinh^{-1}\frac{\sqrt{a}\sqrt{R}}{\sqrt{(b^2-ac)}},$$

or

$$= \frac{1}{\sqrt{(-a)}}\sin^{-1}\frac{\sqrt{(-a)}\sqrt{R}}{\sqrt{(b^2-ac)}},$$

the real form to be chosen according to the signs of a and $ac-b^2$.

EXAMPLES.

1. $\int \frac{dx}{\sqrt{(1+x+x^2)}} = \cosh^{-1}\frac{2\sqrt{(1+x+x^2)}}{\sqrt{3}}$.

2. $\int \frac{dx}{\sqrt{(x^2+2ax\cos\phi+a^2)}} = \cosh^{-1}\frac{\sqrt{(x^2+2ax\cos\phi+a^2)}}{a\sin\phi}$.

3. $\int \frac{dx}{\sqrt{(x^2+2ax\cosh\phi+a^2)}} = \sinh^{-1}\frac{\sqrt{(x^2+2ax\cosh\phi+a^2)}}{a\sinh\phi}$.

4. Prove that, if $\frac{dx}{\sqrt{(ax^2+2bx+c)}} + \frac{dy}{\sqrt{(ay^2+2by+c)}} = 0$,

$$\sqrt{(ax^2+2bx+c)} + \sqrt{(ay^2+2by+c)} = C(x-y).$$

13. Similarly, supposing R positive,

$$\int \frac{dx}{R} = -\frac{1}{\sqrt{(b^2-ac)}}\operatorname{cosech}^{-1}\frac{\sqrt{a}\sqrt{R}}{\sqrt{(b^2-ac)}},$$

$$\text{or} \quad = -\frac{1}{\sqrt{(b^2-ac)}}\operatorname{sech}^{-1}\frac{\sqrt{(-a)}\sqrt{R}}{\sqrt{(b^2-ac)}},$$

$$\text{or} \quad = \frac{1}{\sqrt{(ac-b^2)}}\sec^{-1}\frac{\sqrt{a}\sqrt{R}}{\sqrt{(ac-b^2)}};$$

the real form to be chosen.

EXAMPLES.

1. $\int_x^\infty \frac{dx}{x^2+2ax\cosh\alpha+a^2} = \frac{1}{a\sinh\alpha}\operatorname{cosech}^{-1}\frac{\sqrt{(x^2+2ax\cosh\alpha+a^2)}}{a\sinh\alpha}$.

2. $\int_x^\infty \frac{dx}{x^2+2ax\cos\beta+a^2} = \frac{1}{a\sin\beta}\operatorname{cosec}^{-1}\frac{\sqrt{(x^2+2ax\cos\beta+a^2)}}{a\sin\beta}$.

3. $\int \frac{dx}{x^2+2ax\sinh\alpha-a^2}$. 4. $\int R^{-\frac{3}{2}}dx$, $\int R^{-\frac{5}{2}}dx$, &c.

14. The elliptic integral of the first kind in the canonical form employed by Weierstrass,

$$\int \frac{dx}{\sqrt{(4x^3-g_2x-g_3)}} \quad \text{or} \quad \int \frac{dx}{2\sqrt{(x-e_1.x-e_2.x-e_3)}},$$

when $4x^3-g_2x-g_3$ is resolved into its linear factors, $(x-e_1)$, $(x-e_2)$, $(x-e_3)$, degenerates into a circular or hyperbolic integral, if two of the three roots, e_1, e_2, e_3, become equal.

It is supposed that $e_1>e_2>e_3$; and then (i.) if $e_2=e_3$, the integral

$$\int_{e_1}^x \frac{dx}{2(x-e_3)\sqrt{(x-e_1)}} = \frac{1}{\sqrt{(e_1-e_3)}}\sin^{-1}\sqrt{\frac{x-e_1}{x-e_3}};$$

(ii.) if $e_2=e_1$, the integral

$$\int \frac{dx}{2(x-e_1)\sqrt{(x-e_3)}} = \frac{1}{\sqrt{(e_1-e_3)}}\sinh^{-1}\sqrt{\frac{x-e_3}{e_1-x}} \quad (x<e_1),$$

or $$= \frac{1}{\sqrt{(e_1-e_3)}} \cosh^{-1} \sqrt{\frac{x-e_3}{x-e_1}} \quad (x>e_1).$$

Similarly,

$$\int \frac{dx}{2(e_1-x)\sqrt{(e_3-x)}} = \frac{1}{\sqrt{(e_1-e_3)}} \cos^{-1} \sqrt{\frac{e_3-x}{e_1-x}};$$

$$\int \frac{dx}{2(e_3-x)\sqrt{(e_1-x)}} = \frac{1}{\sqrt{(e_1-e_3)}} \cosh^{-1} \sqrt{\frac{e_3-x}{e_1-x}}, \quad (x<e_3),$$

or $$= \frac{1}{\sqrt{(e_1-e_3)}} \sinh^{-1} \sqrt{\frac{e_3-x}{x-e_1}}, \quad (x>e_3).$$

Examples.

1. $$\int \frac{dx}{\sqrt{(x^2+3x+2)}} = 2\sinh^{-1}\sqrt{(x+1)} = 2\cosh^{-1}\sqrt{(x+2)} = \sinh^{-1} 2\sqrt{(x^2+3x+2)}.$$

2. $$\int \frac{dx}{(1+x)\sqrt{(1+2x)}} = 2\sin^{-1}\sqrt{\frac{1+2x}{2+2x}} = 2\tan^{-1}\sqrt{(1+2x)} = 2\sec^{-1}\sqrt{(2+2x)}.$$

3. $$\int \frac{dx}{(1+2x)\sqrt{(1+x)}} = -\sqrt{2}\sinh^{-1}\sqrt{\frac{2+2x}{-1-2x}}, \quad (-\tfrac{1}{2} > x > -1),$$
$$= \sqrt{2}\cosh^{-1}\sqrt{\frac{2+2x}{1+2x}}, \quad (x > -\tfrac{1}{2}).$$

4. $$\int \frac{dx}{(1+x)\sqrt{(1+x+x^2)}} = \cosh^{-1} \frac{2\sqrt{(1+x+x^2)}}{\sqrt{3}\,(1+x)}.$$

5. $$\int \frac{\sqrt{(1+x+x^2)}}{1+x}\,dx = \tfrac{1}{2}x^2 + \cosh^{-1}\frac{2\sqrt{(1+x+x^2)}}{\sqrt{3}\,(1+x)}.$$

6. $$\int \frac{dx}{\sqrt{(1+e^x+e^{2x})}} = \cosh^{-1}\frac{2\sqrt{(1+e^x+e^{2x})}}{\sqrt{3}\,e^x}.$$

7. $$\int \frac{dx}{(x-\gamma)\sqrt{(x-\alpha.x-\beta)}} = \frac{1}{\sqrt{(\alpha-\gamma.\gamma-\beta)}} \sin^{-1}\frac{2\sqrt{(x-\alpha.x-\beta)}\sqrt{(\alpha-\gamma.\gamma-\beta)}}{(x-\gamma)(\alpha-\beta)},$$
or $$= \frac{1}{\sqrt{(\gamma-\alpha.\gamma-\beta)}} \sinh^{-1}\frac{2\sqrt{(x-\alpha.x-\beta)}\sqrt{(\gamma-\alpha.\gamma-\beta)}}{(x-\gamma)(\alpha-\beta)}.$$

8. $$\int \frac{dx}{(x-\gamma)\sqrt{(\alpha-x.x-\beta)}} = \frac{1}{\sqrt{(\gamma-\alpha.\gamma-\beta)}} \sin^{-1}\frac{2\sqrt{(\alpha-x.x-\beta)}\sqrt{(\gamma-\alpha.\gamma-\beta)}}{(x-\gamma)(\alpha-\beta)},$$
or $$= \frac{1}{\sqrt{(\alpha-\gamma.\gamma-\beta)}} \sinh^{-1}\frac{2\sqrt{(\alpha-x.x-\beta)}\sqrt{(\alpha-\gamma.\gamma-\beta)}}{(x-\gamma)(\alpha-\beta)}.$$

Deduce the value of these integrals when $\gamma = \alpha$ or β.

9. Determine the integrals

$$A = \int_\lambda^\infty \frac{d\lambda}{(a^2+\lambda)^2(c^2+\lambda)^{\frac{1}{2}}} \quad \text{and} \quad C = \int_\lambda^\infty \frac{d\lambda}{(a^2+\lambda)(c^2+\lambda)^{\frac{3}{2}}},$$

required in physical problems in connexion with an ellipsoid of revolution.

The integral $$\int_\lambda^\infty \frac{d\lambda}{(a^2+\lambda)\sqrt{(c^2+\lambda)}} = \frac{2}{\sqrt{(a^2-c^2)}} \cos^{-1}\sqrt{\frac{c^2+\lambda}{a^2+\lambda}},$$

or $$= \frac{2}{\sqrt{(c^2-a^2)}} \cosh^{-1}\sqrt{\frac{c^2+\lambda}{a^2+\lambda}};$$

and differentiation with respect to a^2 or c^2 will give the required results for the above integrals. Thus

$$A = -\frac{\sqrt{(c^2+\lambda)}}{(a^2-c^2)(a^2+\lambda)} + \frac{1}{(a^2-c^2)^{\frac{3}{2}}}\cos^{-1}\sqrt{\frac{c^2+\lambda}{a^2+\lambda}},$$

$$\text{or} \quad = -\frac{\sqrt{(c^2+\lambda)}}{(c^2-a^2)(a^2+\lambda)} + \frac{1}{(c^2-a^2)^{\frac{3}{2}}}\cosh^{-1}\sqrt{\frac{c^2+\lambda}{a^2+\lambda}};$$

$$C = +\frac{2}{(a^2-c^2)\sqrt{(c^2+\lambda)}} - \frac{2}{(a^2-c^2)^{\frac{3}{2}}}\cos^{-1}\sqrt{\frac{c^2+\lambda}{a^2+\lambda}},$$

$$\text{or} \quad = +\frac{2}{(c^2-a^2)\sqrt{(c^2+\lambda)}} - \frac{2}{(c^2-a^2)^{\frac{3}{2}}}\cosh^{-1}\sqrt{\frac{c^2+\lambda}{a^2+\lambda}};$$

so that
$$2A + C = \frac{2}{(a^2+\lambda)\sqrt{(c^2+\lambda)}}.$$

10. $$\int\frac{dx}{(x^2-a^2)\sqrt{(x^2-c^2)}} = \frac{1}{a\sqrt{(c^2-a^2)}}\sin^{-1}\frac{a}{c}\sqrt{\frac{x^2-c^2}{x^2-a^2}},$$

$$\text{or} \quad = \frac{1}{a\sqrt{(a^2-c^2)}}\sinh^{-1}\frac{a}{c}\sqrt{\frac{x^2-c^2}{a^2-x^2}}, \quad (x^2<a^2),$$

$$\text{or} \quad = \frac{1}{a\sqrt{(a^2-c^2)}}\cosh^{-1}\frac{a}{c}\sqrt{\frac{x^2-c^2}{x^2-a^2}}, \quad (x^2>a^2).$$

11. $$\int\frac{d\vartheta}{\cos\vartheta\sqrt{(1+\sin\vartheta)}} \quad (\text{if } \sin\vartheta = x)$$

$$= \int\frac{dx}{(1-x^2)\sqrt{(1+x)}} = \tfrac{1}{2}\int\frac{dx}{(1-x)\sqrt{(1+x)}} + \tfrac{1}{2}\int\frac{dx}{(1+x)^{\frac{3}{2}}}$$

$$= \frac{1}{2\sqrt{2}}\sinh^{-1}\sqrt{\frac{1+x}{1-x}} - \frac{1}{\sqrt{(1+x)}}.$$

12. $$\int\sqrt{\left(\frac{x+a}{x+b}\right)}dx = \sqrt{(x+a.x+b)} + \tfrac{1}{2}(a-b)\sinh^{-1}\frac{\sqrt{(x+a.x+b)}}{\frac{1}{2}(a-b)}.$$

15. Returning again to the integral of § 6, and expressing the result as a function of the argument $y = \dfrac{\sqrt{R}}{x-p}$, where $R = ax^2+2bx+c$,

$$\int\frac{dx}{(x-p)\sqrt{R}} = \frac{1}{\sqrt{(-ap^2-2bp-c)}}\sin^{-1}\frac{\sqrt{R}\sqrt{(-ap^2-2bp-c)}}{(x-p)\sqrt{(b^2-ac)}}$$

$$\text{or} \quad = \frac{1}{\sqrt{(ap^2+2bp+c)}}\sinh^{-1}\frac{\sqrt{R}\sqrt{(ap^2+2bp+c)}}{(x-p)\sqrt{(b^2-ac)}},$$

$$\text{or} \quad = \frac{1}{\sqrt{(ap^2+2bp+c)}}\cosh^{-1}\frac{\sqrt{R}\sqrt{(ap^2+2bp+c)}}{(x-p)\sqrt{(ac-b^2)}},$$

the real form to be chosen for the result.

When p is a root of the equation $R=0$, $ap^2+2bp+c=0$, and the integral is algebraic; and, as a limiting form,

$$\int\frac{dx}{(x-p)\sqrt{R}} = \frac{\sqrt{R}}{(x-p)\sqrt{(b^2-ac)}}.$$

EXAMPLES.

1. $$\int \frac{dx}{\sqrt{\{(x+1)^3(x-1)\}}} = \sqrt{\frac{x-1}{x+1}}.$$

2. $$\int \frac{dx}{x\sqrt{(x^2-2x\cosh\alpha+1)}} = \sinh^{-1}\frac{\sqrt{(x^2-2x\cosh\alpha+1)}}{x\sinh\alpha}.$$

3. $$\int \frac{dx}{x\sqrt{(x^2-2x\cos\alpha+1)}} = \cosh^{-1}\frac{\sqrt{(x^2-2x\cos\alpha+1)}}{x\sin\alpha}.$$

4. $$\int \frac{x\,dx}{(2x+1)\sqrt{\{(1-x)(3x-1)\}}} = \frac{1}{2\sqrt{3}}\sin^{-1}\sqrt{\{3(1-x)(3x-1)\}} - \frac{1}{2\sqrt{15}}\sin^{-1}\frac{\sqrt{\{15(1-x)(3x-1)\}}}{2x+1}.$$

5. $$\int \sqrt{\left\{\frac{1-\cos\vartheta}{(1+\cos\vartheta)\cos\vartheta\,(2+\cos\vartheta)}\right\}}\,d\vartheta \quad (\text{if } \cos\vartheta = x)$$
$$= -\int \frac{dx}{(1+x)\sqrt{(2x+x^2)}} = \cos^{-1}\frac{\sqrt{(2x+x^2)}}{1+x} = \sin^{-1}\frac{1}{1+x} = \operatorname{cosec}^{-1}(1+\cos\vartheta).$$

6. $$\int \frac{\sin^3\vartheta\,d\vartheta}{(1+\cos^2\vartheta)\sqrt{(1+\cos^2\vartheta+\cos^4\vartheta)}} \quad (\text{if } \cos\vartheta = x)$$
$$= -\int \frac{1-x^2}{(1+x^2)\sqrt{(1+x^2+x^4)}}\,dx = \sin^{-1}\frac{\sqrt{(1+x^2+x^4)}}{1+x^2} = \cos^{-1}\frac{x}{1+x^2}.$$

7. $$\int \frac{\sec^2\vartheta\tan^3\vartheta\,d\vartheta}{(\sec^2\vartheta+1)^2\sqrt{(\sec^4\vartheta-\sec^2\vartheta+1)}} \quad (\text{if } \sec^2\vartheta = x)$$
$$= \tfrac{1}{2}\int \frac{x-1}{(x+1)^2\sqrt{(x^2-x+1)}}\,dx = \tfrac{1}{3}\,\frac{\sqrt{(x^2-x+1)}}{x+1}.$$

But $$\int \frac{\sec^2\vartheta\tan\vartheta\,d\vartheta}{(\sec^2\vartheta+1)\sqrt{(\sec^4\vartheta-\sec^2\vartheta+1)}} = \tfrac{1}{2}\int \frac{dx}{(x+1)\sqrt{(x^2-x+1)}} = \frac{1}{2\sqrt{3}}\cosh^{-1}\frac{2\sqrt{(x^2-x+1)}}{x+1}.$$

16. Next, supposing the factor $x-p$ repeated r times in the denominator D, then a corresponding integral

$$\int \frac{dx}{(x-p)^r\sqrt{R}}$$

can be obtained from the appropriate one of the preceding results by differentiating r times with respect to p.

Or, by integration by parts, we can obtain a formula of reduction of the form

$$\int \frac{dx}{(x-p)^r\sqrt{R}} = \frac{A\sqrt{R}}{(x-p)^{r-1}} + B\int \frac{dx}{(x-p)^{r-1}\sqrt{R}} + C\int \frac{dx}{(x-p)^{r-2}\sqrt{R}},$$

and differentiating both sides and equating coefficients, the values of A, B, C are easily determined.

Supposing, however, that the usual substitution $x-p = \dfrac{1}{y}$ had been employed; then this integral would have been made to depend

upon one of the form

$$\int \frac{y^{r-1}\,dy}{\sqrt{(a'y^2+2b'y+c')}},$$

analogous to the integral corresponding to a term of the quotient of N'/D', and finally integrable by a formula of reduction.

The Integral $\int \frac{Mx+N}{Ax^2+2Bx+C}\frac{dx}{\sqrt{R}}$.

17. This is the integral corresponding to a single quadratic factor of D, where $Ax^2+2Bx+C$ may be supposed always positive for all values of x, and therefore A and $AC-B^2$ are positive.

Generally, the integral corresponding to a repeated quadratic factor of D will be of the form

$$\int \frac{\phi(x)}{(Ax^2+2Bx+C)^r}\frac{dx}{\sqrt{R}},$$

which can be deduced from the result of the preceding integral by differentiating any required number of times with respect to A, B, or C.

The constituent substitution required in this integration is

$$y=\sqrt{\left(\frac{ax^2+2bx+c}{Ax^2+2Bx+C}\right)};$$

and the form of the result will depend on the curves which are the *graphs* of this function y of x (Chrystal, *Algebra*, p. 448).

(i.) If $ac-b^2$, and therefore also a, is positive in order that $\sqrt{R}$ should be real, then y cannot vanish, and is real for all real values of x, and therefore the graph is Fig. 1, supposing $Ab-aB$ positive, so that $y^2-\frac{a}{A}$ is positive for large positive values of x.

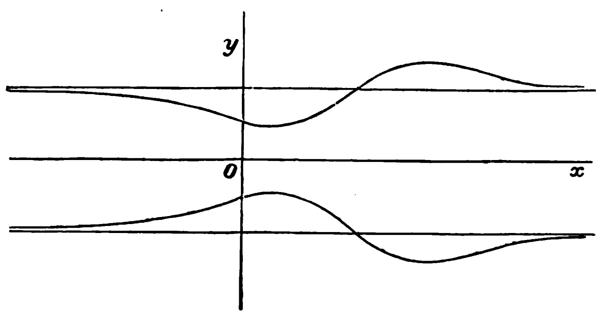

Fig. 1.

(ii.) If $ac-b^2$ is negative, then y can vanish, and x is limited by α and β, the roots of $ax^2+2bx+c=0$; being constrained to lie outside these limits if a is positive, and between the limits if a is negative, the graphs of y being given by Fig. 2.

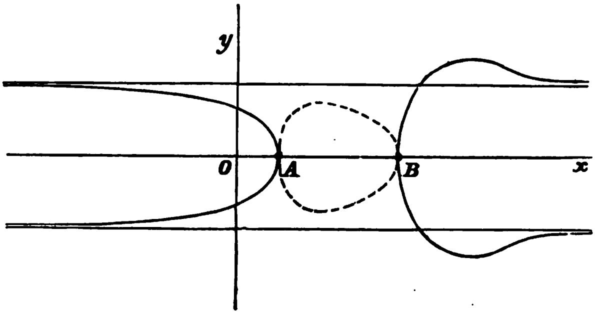

FIG. 2.

Denoting $ax^2+2bx+c$ by R, and $Ax^2+2bx+C$ by U, and differentiating y logarithmically,

$$\frac{1}{y}\frac{dy}{dx}=\frac{ax+b}{R}-\frac{Ax+B}{U}$$

$$=\frac{(aB-Ab)\,x^2+(aC-Ac)\,x+bC-Bc}{UR}=\frac{J}{UR},$$

denoting the Jacobian of U and R by $4J$.

Again, $$y^2-\lambda=\frac{(a-\lambda A)\,x^2+2\,(b-\lambda B)\,x+c-\lambda C}{U}$$

is a perfect square, if

$$(a-\lambda A)(c-\lambda C)-(b-\lambda B)^2=0,$$

or $$(AC-B^2)\,\lambda^2-(aC+Ac-2Bb)\,\lambda+ac-b^2=0;$$

and, denoting the roots of this quadratic in λ by λ_1 and λ_2, then

$$\lambda_1-y^2=\frac{(mx+n)^2}{U},$$

$$y^2-\lambda_2=\frac{(m'x+n')^2}{U},$$

suppose, where $m^2=\lambda_1A-a,\ m'^2=a-\lambda_2A,\ \ldots;$

so that $$J=\sqrt{(AC-B^2)}(mx+n)(m'x+n');$$

and the *turning-points* of y are given by $mx+n=0$, and $m'x+n'=0$.

Since $$\frac{dx}{dy}=\frac{UR}{Jy}=\frac{U^{\frac{1}{2}}R^{\frac{1}{2}}}{J},$$
therefore the integral considered
$$\int\frac{Mx+N}{UR^{\frac{1}{2}}}\,dx=\int\frac{Mx+N}{J}\,U^{\frac{1}{2}}\,dy$$
$$=\int\frac{P(m'x'+n')+P'(mx+n)}{(mx+n)(m'x+n')}\,dy\left\{\frac{mx+n}{\sqrt{(\lambda_1-y^2)}}\text{ or }\frac{m'x+n'}{\sqrt{(y^2-\lambda_2)}}\right\},$$
$$=P\int\frac{dy}{\sqrt{(\lambda_1-y^2)}}+P'\int\frac{dy}{\sqrt{(y^2-\lambda_2)}},$$
putting, identically,
$$P(m'x+n')+P'(mx+n)\equiv\sqrt{(AC-B^2)}(Mx+N),$$
so that, making $mx+n$ and $m'x+n'$ alternately zero,
$$P=\sqrt{(AC-B^2)}\,\frac{mN-Mn}{mn'-m'n},\qquad P'=\sqrt{(AC-B^2)}\,\frac{Mn'-m'N}{mn'-m'n}.$$
(i.) If $ac-b^2$ is positive, then λ_1 and λ_2 are positive; and the integral
$$=-P\cos^{-1}\frac{y}{\sqrt{\lambda_1}}+P'\cosh^{-1}\frac{y}{\sqrt{\lambda_2}}$$
$$=-P\cos^{-1}\sqrt{\left(\frac{1}{\lambda_1}\,\frac{ax^2+2bx+c}{Ax^2+2Bx+C}\right)}+P'\cosh^{-1}\sqrt{\left(\frac{1}{\lambda_2}\,\frac{Ax^2+2Bx+C}{ax^2+2bx+c}\right)};$$
(ii.) if $ac-b^2$ is negative, then λ_2 is negative; and the integral
$$=P\sin^{-1}\frac{y}{\sqrt{\lambda_1}}+P'\sinh^{-1}\frac{y}{\sqrt{(-\lambda_2)}}$$
$$=P\sin^{-1}\sqrt{\left(\frac{1}{\lambda_1}\,\frac{ax^2+2bx+c}{Ax^2+2Bx+C}\right)}+P'\sinh^{-1}\sqrt{\left(-\frac{1}{\lambda_2}\,\frac{Ax^2+2Bx+C}{ax^2+2bx+c}\right)};$$
thus giving the value of the general integral
$$\int\frac{Mx+N}{(Ax^2+2Bx+C)\sqrt{(ax^2+2bx+c)}}\,dx,$$
according as $ac-b^2$ is positive or negative, in terms of functions of the quotient of the constituents $ax^2+2bx+c$ and $Ax^2+2Bx+C$.

The differentiation of this integral with respect to A, B, or C will enable us to deduce the value of the integral
$$\int\frac{\phi(x)\,dx}{(Ax^2+2Bx+C)^r\sqrt{(ax^2+2bx+c)}},$$
where $\phi(x)$ denotes any rational integral function of x, of degree lower than $2r$; or a formula of reduction may be employed.

Construction of Numerical Examples.

18. The following method of constructing numerical examples of the preceding integrations is due to Mr. Robert Russell, M.A., of Trinity College, Dublin.

The quadratics

$$J = (aB-Ab)\,x^2+(aC-Ac)\,x+bC-Bc = 0,$$

and
$$(AC-B^2)\,\lambda^2-(aC+Ac-2bB)\,\lambda+ac-b^2 = 0,$$

are by a homographic substitution transformable into each other, the substitution being

$$\lambda = \frac{ax+b}{Ax+B};$$

and, consequently, if the roots of one quadratic are rational, so are also the roots of the other.

Denoting by t_1 and t_2 the roots of $J=0$, then

$$\lambda_1\,(Ax^2+2Bx+C)-(ax^2+2bx+c) = (\lambda_1A-a)(x-t_1)^2,$$

$$\lambda_2\,(Ax^2+2Bx+C)-(ax^2+2bx+c) = (\lambda_2A-a)(x-t_2)^2,$$

so that

$$Ax^2+2Bx+C = p\,(x-t_1)^2+q\,(x-t_2)^2,$$

$$ax^2+2\,bx+\;c = p'\,(x-t_1)^2+q'(x-t_2)^2,$$

where
$$\lambda_1 = q'/q,\quad \lambda_2 = p'/p.$$

We may now assign any arbitrary integral values to t_1, t_2, p, q, p', q', and then integral values of A, B, C, a, b, c result.

Since $Ax^2+2Bx+C$ is supposed positive for all real values of x, therefore p and q must have positive values; but, according as $ac-b^2$ is positive or negative, so p' and q' must be taken positive, or one of them, say q', negative.

For example, take $t_1 = 3$, $t_2 = 4$; and

(i.) $p = 25$, $q = 49$, $p' = 81$, $q' = 64$, then

$$Ax^2+2Bx+C = 25\,(x-3)^2+49\,(x-4)^2 = 74x^2-542x+1009,$$

$$ax^2+2bx+c = 81\,(x-3)^2+64\,(x-4)^2 = 145x^2-998x+1753,$$

and
$$\int\frac{P(x-3)+P'(x-4)}{(74x^2-542x+1009)\,\sqrt{(145x^2-998x+1753)}}\,dx = \frac{1}{\sqrt{(2369)}}\times$$

$$\left\{\tfrac{1}{7}P\cos^{-1}\tfrac{8}{9}\sqrt{\left(\frac{145x^2-998x+1753}{74x^2-542x+1009}\right)}-\tfrac{1}{7}P'\cosh^{-1}\tfrac{7}{8}\sqrt{\left(\frac{145x^2-998x+1753}{74x^2-542x+1009}\right)}\right\}.$$

(ii.) $p = 25,\ q = 49,\ p' = 81,\ q' = -64,$

$$Ax^2+2Bx+C = 74x^2-542x+1009, \quad ax^2+2bx+c = 17x^2+26x-295,$$

$$\int \frac{P(x-3)+P'(x-4)}{(74x^2-542x+1009)\sqrt{(17x^2+26x-295)}}\,dx = \frac{1}{\sqrt{(5569)}} \times$$

$$\left\{-\tfrac{1}{5}P\sin^{-1}\tfrac{5}{8}\sqrt{\frac{17x^2+26x-295}{74x^2-542x+1009}} - \tfrac{1}{7}P'\sinh^{-1}\tfrac{7}{8}\sqrt{\frac{17x^2+26x-295}{74x^2-542x+1009}}\right\}.$$

Examples.

1. $$\int \frac{x-1}{(3x^2-10x+9)\sqrt{(5x^2-16x+14)}}\,dx = \cos^{-1}\sqrt{\frac{1}{2}\,\frac{5x^2-16x+14}{3x^2-10x+9}}.$$

2. $$\int \frac{x-2}{(3x^2-10x+9)\sqrt{(5x^2-16x+14)}}\,dx = \cosh^{-1}\sqrt{\frac{2}{3}\,\frac{5x^2-16x+14}{3x^2-10x+9}}.$$

3. $$\int \frac{x-1}{(5x^2-18x+17)\sqrt{(6x^2-22x+21)}}\,dx = \cosh^{-1}\sqrt{\frac{6x^2-22x+21}{5x^2-18x+17}}.$$

4. $$\int \frac{x-2}{(5x^2-18x+17)\sqrt{(6x^2-22x+21)}}\,dx = \cos^{-1}\sqrt{\frac{4}{5}\,\frac{6x^2-22x+21}{5x^2-18x+17}}.$$

5. $$\int \frac{x-2}{(3x^2-10x+9)\sqrt{(x^2-8x+10)}}\,dx = \frac{1}{\sqrt{14}}\sin^{-1}\sqrt{\frac{2}{3}\,\frac{x^2-8x+10}{3x^2-10x+9}}.$$

6. $$\int \frac{x-1}{(3x^2-10x+9)\sqrt{(x^2-8x+10)}}\,dx = \frac{1}{\sqrt{14}}\sinh^{-1}\sqrt{\frac{1}{2}\,\frac{x^2-8x+10}{3x^2-10x+9}}.$$

7. $$\int \frac{dx}{(Ax^2+C)\sqrt{(ax^2+c)}} = \frac{1}{\sqrt{C}\sqrt{(Ac-aC)}}\cos^{-1}\sqrt{\frac{C}{c}\,\frac{ax^2+c}{Ax^2+C}},$$

$$\text{or} \quad = \frac{1}{\sqrt{C}\sqrt{(aC-Ac)}}\cosh^{-1}\sqrt{\frac{C}{c}\,\frac{ax^2+c}{Ax^2+C}},$$

$$\text{or} \quad = \frac{1}{\sqrt{C}\sqrt{(aC-Ac)}}\sinh^{-1}\sqrt{-\frac{C}{c}\,\frac{ax^2+c}{Ax^2+C}}.$$

If $Ac-aC=0$, then the argument $\dfrac{ax^2+c}{Ax^2+C}$ reduces to a constant, and the integral is algebraic. Writing the integral in the form

$$\int \frac{dx}{(Ax^2+C)\sqrt{(ax^2+c)}} = \frac{1}{\sqrt{C}\sqrt{(Ac-aC)}}\sin^{-1}\frac{x\sqrt{(Ac-aC)}}{\sqrt{\{c(Ax^2+C)\}}},$$

$$\text{or} \quad = \frac{1}{\sqrt{C}\sqrt{(aC-Ac)}}\sinh^{-1}\frac{x\sqrt{(aC-Ac)}}{\sqrt{\{c(Ax^2+C)\}}};$$

then, when $Ac-aC=0$, the limiting value of the integral is seen to be

$$\frac{x}{\sqrt{Cc}\sqrt{(Ax^2+C)}} = \frac{x}{C\sqrt{(ax^2+c)}}.$$

8. $$\int \frac{dx}{(a^2-\tan^2 x)\sqrt{(b^2-\tan^2)\,x}}, \text{ if } \tan x = y,$$

$$= \int \frac{dy}{(1+y^2)(a^2-y^2)\sqrt{(b^2-y^2)}} = \frac{1}{1+a^2}\int\left(\frac{1}{1+y^2}+\frac{1}{a^2-y^2}\right)\frac{dy}{\sqrt{(b^2-y^2)}};$$

and
$$\frac{dy}{(1+y^2)\sqrt{(b^2-y^2)}}=\frac{1}{\sqrt{(1+b^2)}}\cos^{-1}\sqrt{\frac{b^2-y^2}{b^2+b^2y^2}};$$
$$\int\frac{dy}{(a^2-y^2)\sqrt{(b^2-y^2)}}=\frac{1}{\sqrt{(a^2-b^2)}}\cos^{-1}\sqrt{\frac{a^2}{b^2}\,\frac{b^2-\cancel{x}^2}{a^2-y^2}},\quad y^2$$
a

or
$$=\frac{1}{\sqrt{(b^2-a^2)}}\cosh^{-1}\sqrt{\frac{a^2}{b^2}\,\frac{b^2-y^2}{a^2-y^2}}.$$
a

19. The integrals (i.) $\displaystyle\int\frac{M\cos\vartheta+N}{A\cos^2\vartheta+2B\cos\vartheta+C}\,d\vartheta,$

(ii.) $\displaystyle\int\frac{M\cosh u+N}{A\cosh^2 u+2B\cosh u+C}\,du,$

(iii.) $\displaystyle\int\frac{M\sinh v+N}{A\sinh^2 v+2B\sinh v+C}\,dv,$

are reduced to the preceding form of § 17, by putting $x=\cos\vartheta$, $\cosh u$, or $\sinh v$; and therefore the integrals are of the form

(i.)
$$P\sin^{-1}\sqrt{\frac{k\sin^2\vartheta}{A\cos^2\vartheta+2B\cos\vartheta+C}}+P'\sinh^{-1}\sqrt{\frac{-k'\sin^2\vartheta}{A\cos^2\vartheta+2B\cos\vartheta+C}},$$

(ii.)
$$P\sin^{-1}\sqrt{\frac{k\sinh^2 u}{A\cosh^2 u+2B\cosh u+C}}$$
$$+P'\sinh^{-1}\sqrt{\frac{-k'\sinh^2 u}{A\cosh^2 u+2B\cosh u+C}},$$

(iii.)
$$-P\cos^{-1}\sqrt{\frac{k\cosh^2 v}{A\sinh^2 v+2B\sinh v+C}}$$
$$+P'\cosh^{-1}\sqrt{\frac{k'\cosh^2 v}{A\sinh^2 v+2B\sinh v+C}}.$$

When $\cos\vartheta=x$, then $R=1-x^2$, so that now $t_1t_2=1$, and we may put
$$R=\frac{(x-t_1)^2-(t_1x-1)^2}{t_1^2-1}.$$

We may now put
$$Ax^2+2Bx+C=p\,(x-t_1)^2+q\,(t_1x-1)^2,$$
and then assign any arbitrary numerical values to t_1, p and q.

When $\cosh u=x$, then R is of the form x^2-1, so that
$$R=\frac{(x-t_1)^2-(t_1x-1)^2}{1-t_1^2},$$
and
$$Ax^2+2Bx+C=p\,(x-t_1)^2+q\,(t_1x-1)^2.$$

Finally, when $\sinh u = x$, then R is of the form x^2+1, so that

$$R = \frac{(x-t_1)^2+(t_1x+1)^2}{1+t_1^2},$$

$$Ax^2+2Bx+C = p\,(x-t_1)^2+q\,(t_1x+1)^2,$$

where t_1, p and q may have any arbitrary numerical values.

EXAMPLES.

1. $\displaystyle\int \frac{3-\cos\vartheta}{5-6\cos\vartheta+5\cos^2\vartheta}\,d\vartheta = \tfrac{1}{2}\sin^{-1}\frac{2\sin\vartheta}{\sqrt{(5-6\cos\vartheta+5\cos^2\vartheta)}}.$

2. $\displaystyle\int \frac{1-3\cos\vartheta}{5-6\cos\vartheta+5\cos^2\vartheta}\,d\vartheta = \tfrac{1}{2}\sinh^{-1}\frac{2\sin\vartheta}{\sqrt{(5-6\cos\vartheta+5\cos^2\vartheta)}},$

whence $$\int \frac{M+N\cos\vartheta}{5-6\cos\vartheta+5\cos^2\vartheta}\,d\vartheta.$$

3. $\displaystyle\int \frac{\cosh u-2}{5\cosh^2 u-6\cosh u+5}\,du = -\frac{1}{\sqrt{6}}\sinh^{-1}\frac{\sqrt{3}\sinh u}{\sqrt{(5\cosh^2 u-8\cosh u+5)}}.$

4. $\displaystyle\int \frac{2\cosh u-1}{5\cosh^2 u-6\cosh u+5}\,du = -\frac{1}{\sqrt{6}}\sin^{-1}\frac{\sqrt{3}\sinh u}{\sqrt{(5\cosh^2 u-8\cosh u+5)}},$

whence $$\int \frac{M\cosh u+N}{5\cosh^2 u-6\cosh u+5}\,du.$$

5. Determine $\displaystyle\int \frac{\sinh v-2}{6\sinh^2 v-4\sinh v+9}\,dv$, and $\displaystyle\int \frac{2\sinh v+1}{6\sinh^2 v-4\sinh v+9}\,dv$;

and generally $$\int \frac{M\sinh v+N}{6\sinh^2 v-4\sinh v+9}\,dv.$$

6. Find $\displaystyle\int \frac{\sinh v\pm 1}{\sinh^2 v-\sinh v+1}\,dv$, and thence generally $\displaystyle\int \frac{M\sinh v+N}{\sinh^2 v-\sinh v+1}\,dv.$

Analogies and Properties of the Hyperbolic Functions.

20. The preceding integrations will illustrate the use of the hyperbolic functions, direct and inverse, in conjunction with the circular functions; and the following geometrical definitions will further elucidate the relations between them.

Taking two rectangular axes Ox and Oy, and describing a circle AP of radius a with centre O, the equation of which is therefore

$$x^2+y^2 = a^2,$$

then if the circular measure of the angle AOP is denoted by ϑ, it follows from the definitions of ordinary trigonometry that

$$\text{the arc } AP = a\vartheta, \quad \text{the sector } AOP = \tfrac{1}{2}a^2\vartheta;$$

and if the ordinate PM and the tangent PN is drawn, then

$$OM = a\cos\vartheta,\quad MP = a\sin\vartheta,\quad PN = a\tan\vartheta,\quad ON = a\sec\vartheta,$$
$$AM = a\,\mathrm{vers}\,\vartheta.$$

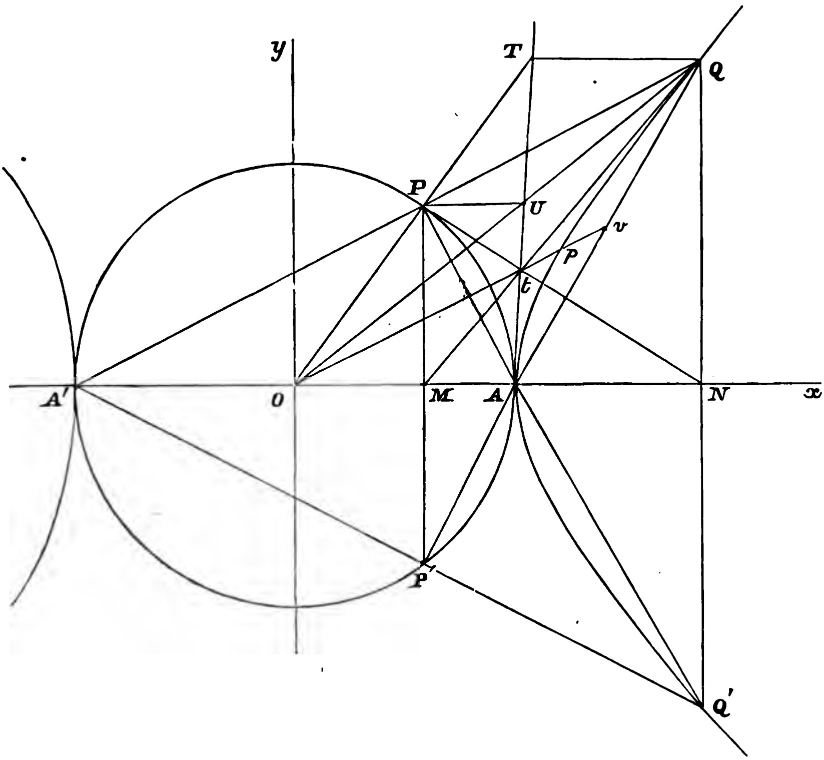

FIG. 3.

Incidentally we notice that the triangle $OAP = \frac{1}{2}a^2\sin\vartheta$, the sector $OAP = \frac{1}{2}a^2\vartheta$, and the triangle $ONP = \frac{1}{2}a^2\tan\vartheta$; and these are obviously in ascending order of magnitude, proving that $\sin\vartheta$, ϑ, and $\tan\vartheta$ are in ascending order of magnitude in the first quadrant.

Now, if the ordinate NQ is erected, of length equal to the tangent NP, then the coordinates x and y of Q are given by

$$x = ON = a\sec\vartheta,\quad y = NQ = a\tan\vartheta;$$

so that the equation of the locus of Q is

$$x^2 - y^2 = a^2,$$

and Q describes a rectangular hyperbola.

21. But we may also express the coordinates of Q by

$$x = a\cosh u, \quad y = a\sinh u,$$

so that

$$\cosh u = \sec\vartheta, \quad \sinh u = \tan\vartheta,$$

and then we shall find that the sectorial area AOQ, bounded by the vectors OA and OQ, and the hyperbolic arc AQ, is equal to $\frac{1}{2}a^2u$.

Also $\tanh u$, u, and $\sinh u$ are in ascending order of magnitude.

When u and ϑ are connected by this relation, then

$$e^u = \cosh u + \sinh u = \sec\vartheta + \tan\vartheta,$$

or

$$u = \log(\sec\vartheta + \tan\vartheta) = \log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\vartheta).$$

Conversely, ϑ is then called by Cayley (*Elliptic Functions*, § 79) the *Gudermannian function* of u, and denoted by $\operatorname{gd} u$; so that, if

$$\vartheta = \operatorname{gd} u,$$

then

$$u = \log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\vartheta) = \operatorname{gd}^{-1}\vartheta,$$

and

$$\frac{du}{d\vartheta} = \frac{d\operatorname{gd}^{-1}\vartheta}{d\vartheta} = \sec\vartheta.$$

The Gudermannian function connects the circular and hyperbolic functions; so that the table given by Prof. Cayley in the *Quarterly Journal of Mathematics*, Vol. xx., p. 213, extracted from Legendre's *Fonctions Elliptiques*, t. ii., Table iv., connecting u and ϑ (in degrees), enables us to determine the numerical values of the hyperbolic functions of the argument u, by means of the above relations—

$$\sinh u = \tan\vartheta, \quad \cosh u = \sec\vartheta, \ \&c.$$

Tables of the hyperbolic functions have been calculated by Lambert, Gudermann, Gronan, Forti, &c. (*Die hyperbolischen Functionen*, von Prof. Dr. E. Heis. Halle, 1875).

22. Since $\cos iu = \cosh u$, we notice that, given $\vartheta = \operatorname{gd} u$, or $\cos\vartheta\cosh u = 1$, then $\cosh i\vartheta\cos iu = 1$ or $iu = \operatorname{gd} i\vartheta$; so that, if the expansion of ϑ in ascending powers of u is

$$\vartheta = a_1 u - a_3 u^3 + a_5 u^5 - \ldots,$$

involving of necessity only odd powers of u, because ϑ is an odd function of u, the curious example of *reversion of series* follows, that

$$u = a_1\vartheta + a_3\vartheta^3 + a_5\vartheta^5 + \ldots.$$

Since

$$u = \log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\vartheta),$$

therefore

$$\frac{du}{d\vartheta} = \sec\vartheta;$$

and supposing $\sec\vartheta$ expanded in the form

$$\sec\vartheta = 1 + \frac{x^2}{2!}E_2 + \frac{x^4}{4!}E_4 + \ldots + \frac{x^{2n}}{(2n)!}E_{2n} + \ldots,$$

where $E_2, E_4, \ldots E_{2n}, \ldots$ are called Euler's numbers, it follows by integration that

$$a_1 = 1, \quad a_3 = \frac{E_2}{3!}, \quad a_5 = \frac{E_4}{5!}, \ldots a_{2n+1} = \frac{E_{2n}}{(2n+1)!};$$

also $$E_{2n} = \frac{2\,(2n)!}{(\frac{1}{2}\pi)^{2n+1}}\left(1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \ldots\right).$$

23. The relation connecting ϑ and u, in the form

$$\cos\vartheta \cosh u = 1,$$

leads immediately to

$$\tan^2 \tfrac{1}{2}\vartheta = \frac{1-\cos\vartheta}{1+\cos\vartheta} = \frac{\cosh u - 1}{\cosh u + 1} = \tanh^2 \tfrac{1}{2}u;$$

which proves that the line Otp, drawn to t the point of intersection of the tangents at A and P of the circle, and therefore the bisector of the circular sector OAP, is also the bisector of the hyperbolic sector OAQ; also MQ is the tangent of the hyperbola at Q, and passes through t, while At produced meets OP in a point U, such that $AU = MP$, as may easily be demonstrated.

For developments of this subject, the reader is referred to M. Laurent's "Essai sur les Fonctions hyperboliques," in the *Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux*, t. x., 1875. Here ϑ is called the *hyperbolic amplitude* of u, with the notation

$$\vartheta = \operatorname{amh} u;$$

analogous to Jacobi's elliptic amplitude function am u, defined by

$$u = \int_0^{\operatorname{am} u} \frac{d\vartheta}{\sqrt{(1-k^2\sin^2\vartheta)}},$$

which reduces to amh u or gd u, when the modulus $k = 1$.

We notice, as additional anologies, that

$$AT = PN = a\tan\vartheta = NQ = a\sinh u,$$
$$AU = a\tanh u = MP = a\sin\vartheta.$$

Also $$\sin iv = i\sinh v, \quad \cos iv = \cosh v, \quad \tan iv = i\tanh v;$$
$$\sinh iu = -i\sin u, \quad \cosh iu = \cos u, \quad \tanh iu = -i\tan u;$$

so that
$$\sin(u+iv) = \sin u\cosh v + i\cos u\sinh v,$$
$$\sinh(v+iu) = \sinh v\cos u + i\cosh v\sin u,$$
$$\cos(u+iv) = \cos u\cosh v - i\sin u\sinh v,$$
$$\cosh(v+iu) = \cos u\cosh v + i\sin u\sinh v,$$
$$\tan\tfrac{1}{2}(u+iv) = \frac{\sin u + i\sinh v}{\cos u + \cosh v}.$$

Since
$$\cosh u+\sinh u=\exp u,$$
$$\cosh u-\sinh u=\exp(-u),$$
therefore
$$\begin{aligned}\cosh(u+v)&=\tfrac{1}{2}\exp(u+v)+\tfrac{1}{2}\exp(-u-v)\\&=\tfrac{1}{2}(\cosh u+\sinh u)(\cosh v+\sinh v)\\&\quad+\tfrac{1}{2}(\cosh u-\sinh u)(\cosh v-\sinh v)\\&=\cosh u\cosh v+\sinh u\sinh v;\end{aligned}$$
similarly
$$\sinh(u+v)=\sinh u\cosh v+\cosh u\sinh v,$$
$$\tanh(u+v)=\frac{\tanh u+\tanh v}{1+\tanh u\tanh v};$$
$$\sinh u+\sinh v=2\sinh\tfrac{1}{2}(u+v)\cosh\tfrac{1}{2}(u-v),$$
$$\sinh u-\sinh v=2\cosh\tfrac{1}{2}(u+v)\sinh\tfrac{1}{2}(u-v),$$
$$\cosh u+\cosh v=2\cosh\tfrac{1}{2}(u+v)\cosh\tfrac{1}{2}(u-v),$$
$$\cosh u-\cosh v=2\sinh\tfrac{1}{2}(u+v)\sinh\tfrac{1}{2}(u-v).$$

The hyperbolic functions have an imaginary period, $2i\pi$ for the hyperbolic sine and cosine, and $i\pi$ for the hyperbolic tangent; and
$$\sinh\tfrac{1}{2}i\pi=i,\quad\cosh\tfrac{1}{2}i\pi=0,\quad\tanh\tfrac{1}{2}i\pi=\infty;$$
$$\sinh i\pi=0,\quad\cosh i\pi=-1,\quad\tanh i\pi=0;$$
$$\sinh 2i\pi=0,\quad\cosh 2i\pi=1,\quad\tanh 2i\pi=0;$$
$$\sinh(\tfrac{1}{2}i\pi+v)=i\cosh v,\qquad\cosh(\tfrac{1}{2}i\pi+v)=i\sinh v,$$
$$\tanh(\tfrac{1}{2}i\pi+v)=\coth v;$$
$$\sinh(i\pi+v)=-\sinh v,\qquad\cosh(i\pi+v)=-\cosh v,$$
$$\tanh(i\pi+v)=\tanh v;$$
$$\sinh(2i\pi+v)=\sinh v,\qquad\cosh(2i\pi+v)=\cosh v,$$
$$\tanh(2i\pi+v)=\tanh v.$$

Analogous to De Moivre's theorem for the circular functions, we have
$$\begin{aligned}(\cosh v\pm\sinh v)^m=\{\exp(\pm v)\}^m&=\exp(\pm mv)\\&=\cosh mv\pm\sinh mv.\end{aligned}$$

Examples on the Hyperbolic Functions.

1. Construct a Table exhibiting the value of any direct hyperbolic function in terms of any other.

2. Construct a similar Table for the inverse hyperbolic functions.

3. Expand, in ascending powers of x, $\cosh x$, $\sinh x$, $\tanh x$, $\sinh^{-1}x$, $\tanh^{-1}x$.

4. Prove that

$$\sin \text{ or } \tan \left(a\frac{d}{dx}\right) \frac{\sin}{\cos} mx = \sinh \quad \text{or} \tanh \quad am \frac{\cos}{-\sin} mx,$$

$$\sin^{-1} \text{ or } \tan^{-1} \left(a\frac{d}{dx}\right) \frac{\sin}{\cos} mx = \sinh^{-1} \text{ or } \tanh^{-1} am \frac{\cos}{-\sin} mx.$$

5. Prove that, in the ordinary catenary,

$$\frac{y}{c} = \cosh\frac{x}{c}, \quad \frac{s}{c} = \sinh\frac{x}{c}.$$

6. Prove that, in the catenary of equal strength,

$$\exp\frac{y}{c}\cos\frac{x}{c} = 1, \quad \frac{x}{c} = \operatorname{gd}\frac{s}{c}.$$

Prove also that the line density is proportional to $\cosh s/c$; and thence show that a piece of flexible cloth, cut out from a plane sheet by the ordinary catenaries $y/c = \pm\cosh x/c$, can hang in a catenary of equal strength.

7. Show that, if $$ix + y = c\log(\alpha + i\beta),$$
then $$\exp y/c \cos x/c = \alpha, \quad \exp y/c \sin x/c = \beta,$$
so that the curves α = constant, and β = constant, are orthogonal catenaries of equa strength; and $\exp y/c \cos(x/c - \gamma)$ = constant is the equation of an oblique trajectory, also a catenary of equal strength, cutting the orthogonal system at an angle γ.

8. Prove that, if $$x + iy = c\cosh(\xi + i\eta),$$
the curves η = constant, and ξ = constant, are confocal ellipses and hyperbolas; and show that the focal distances of a point are $c(\cosh\eta \pm \cos\xi)$.

Prove that the distance between two points defined by (ξ, η), (ξ', η') is

$$c\sqrt{\{\cosh(\eta+\eta') - \cos(\xi+\xi')\}\{\cosh(\eta-\eta') - \cos(\xi-\xi')\}},$$

the same as the distance between the corresponding points (ξ, η'), (ξ', η); and prove that the angle between these joining lines is

$$2\tan^{-1}\frac{\tan\frac{1}{2}(\xi-\xi')}{\tanh\frac{1}{2}(\eta-\eta')}.$$

9. Prove that the equation

$$x + iy = c\tan\tfrac{1}{2}(\phi + i\rho)$$

represents for constant values of ρ and ϕ a system of dipolar circles; which circles are the stereographic projections of a system of parallels and meridians on a sphere.

Draw the stereographic projections of the E. and W. hemispheres of the terrestrial globe, for poles of projection on the equator in longitude 110° W. and 70° E.; and prove that the latitude $\vartheta = \operatorname{gd}\rho$, while ϕ is the longitude.

10. Prove that the equation $\vartheta = \operatorname{gd} n\phi$, connecting ϑ the latitude and ϕ the longitude, is the equation of a *loxodrome* on the sphere, cutting the meridians at a constant angle α, where $n = \cot\alpha$.

11. Prove that, on Mercator's chart, $\vartheta = \operatorname{gd} y/a$, $\phi = x/a$, where ϑ, ϕ denote the latitude and longitude (in circular measure) of a point whose coordinates are x, y on the chart.

Prove that the equation of the curve on the chart, which represents a small circle on the sphere, is of the form

$$\cosh(y-b)/a = \sec\beta\cos x/a,$$
or
$$\sinh(y-b)/a = \tan\beta\cos x/a.$$

12. Show that Cotes's spirals are all included under one of the forms

$$r \sin n\vartheta = a, \quad r n\vartheta = a, \quad r \sinh n\vartheta = a, \quad r \exp n\vartheta = a, \quad r \cosh n\vartheta = a.$$

13. Prove that in the catenary formed by an elastic rope,

$$x/c = u + k \sinh u, \quad y/c = \cosh u + \tfrac{1}{2}k \cosh^2 u, \quad s/c = \tfrac{1}{2}ku + \sinh u + \tfrac{1}{4}k \sinh 2u,$$

putting $\dfrac{dy}{dx} = \sinh u.$ (Laisant.)

The Integrals $\int \sec \vartheta \, d\vartheta$, $\int \operatorname{cosec} \vartheta \, d\vartheta$, $\int \operatorname{sech} u \, du$, $\int \operatorname{cosech} v \, dv$.

24. Suppose the integral of $\sec \vartheta$ is required; according to the method of the ordinary treatises,

$$\int \sec \vartheta \, d\vartheta = \log \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\vartheta) = \operatorname{gd}^{-1} \vartheta.$$

But in order to express the integral as a function of $\sec \vartheta$, put $\sec \vartheta = y$; then

$$\int \sec \vartheta d\vartheta = \int \frac{dy}{\sqrt{(y^2-1)}} = \cosh^{-1} y = \cosh^{-1} \sec \vartheta;$$

and similarly

$$\int \operatorname{sech} u \, du = \cos^{-1} \operatorname{sech} u.$$

Similarly $\int \operatorname{cosec} \vartheta \, d\vartheta$, usually expressed in the form $\log \tan \tfrac{1}{2}\vartheta$, when expressed as a function of $\operatorname{cosec} \vartheta$, leads to

$$\int \operatorname{cosec} \vartheta \, d\vartheta = -\cosh^{-} (\operatorname{cosec} \vartheta),$$

or, as a corrected integral,

$$\int_{\vartheta}^{\frac{1}{2}\pi} \operatorname{cosec} \vartheta \, d\vartheta = \cosh^{-1} (\operatorname{cosec} \vartheta);$$

and

$$\int_{u}^{\infty} \operatorname{cosec} u \, du = \sinh^{-1} (\operatorname{cosech} u).$$

Generally

$$\int \frac{dx}{b \cos x + c \sin x} = \frac{1}{\sqrt{(b^2+c^2)}} \cosh^{-1} \frac{\sqrt{(b^2+c^2)}}{b \cos x + c \sin x}$$

$$= \frac{1}{\sqrt{(b^2+c^2)}} \sinh^{-1} \frac{b \sin x - c \cos x}{b \cos x + c \sin x},$$

and

$$\int \frac{dx}{b \cosh x + c \sinh x} = \frac{1}{\sqrt{(b^2-c^2)}} \cos^{-1} \frac{\sqrt{(b^2-c^2)}}{b \cosh x + c \sinh x}$$

$$= \frac{1}{\sqrt{(b^2-c^2)}} \sin^{-1} \frac{b \sinh x + c \cosh x}{b \cosh x + c \sinh x},$$

or

$$= \frac{1}{\sqrt{(c^2-b^2)}} \sinh^{-1} \frac{\sqrt{(c^2-b^2)}}{b \cosh x + c \sinh x}$$

$$= \frac{1}{\sqrt{(c^2-b^2)}} \cosh^{-1} \frac{b \sinh x + c \cosh x}{b \cosh x + c \sinh x}.$$

EXAMPLES.

1. $\displaystyle\int\frac{dx}{\cos x+\sin x}=\frac{1}{\sqrt{2}}\cosh^{-1}\frac{\sqrt{2}}{\cos x+\sin x}=\frac{1}{\sqrt{2}}\sinh^{-1}\frac{\cos x-\sin x}{\cos x+\sin x}.$

2. $$\int(\tan\tfrac{1}{2}x)^n(\operatorname{cosec}x)^3\,dx \text{ (putting } \tan\tfrac{1}{2}x=y)$$

$$=\int y^n\left(\frac{1+y^2}{2y}\right)^3\frac{2dy}{1+y^2}=\tfrac{1}{4}\int(y^{n-3}+2y^{n-1}+y^{n+1})\,dy=\tfrac{1}{4}\left(\frac{y^{n-2}}{n+2}+\frac{2y^n}{n}+\frac{y^{n+2}}{n+2}\right);$$

and similarly $$\int(\cot\tfrac{1}{2}x)^n(\operatorname{cosec}x)^3\,dx$$

$$=\int\frac{1}{y^n}\left(\frac{1+y^2}{2y}\right)^3\frac{2dy}{1+y^2}=\tfrac{1}{4}\int(y^{-n-3}+2y^{-n-1}+y^{-n+1})\,dy$$

$$=\tfrac{1}{4}\left(\frac{y^{-n+2}}{-n-2}+\frac{2y^{-n}}{-n}+\frac{y^{-n+2}}{-n+2}\right).$$

3. $$\int\frac{d\vartheta}{a+b\tan\vartheta}=\int\frac{\cos\vartheta\,d\vartheta}{a\cos\vartheta+b\sin\vartheta}=M\vartheta+N\log(a\cos\vartheta+b\sin\vartheta),$$

where $$M=\frac{a}{a^2+b^2},\quad N=\frac{b}{a^2+b^2}.$$

4. $$\int\frac{c\cos\vartheta+e\sin\vartheta}{a\cos\vartheta+b\sin\vartheta}=M\vartheta+N\log(a\cos\vartheta+b\sin\vartheta),$$

where $$M=\frac{ac+be}{a^2+b^2},\quad N=\frac{be-ae}{a^2+b^2}.$$

5. $\displaystyle\int\sqrt{(2\cot x)}\,dx=\sin^{-1}\sqrt{(\sin 2x)}+\sinh^{-1}\sqrt{(\sin 2x)}.$

6. $\displaystyle\int\sqrt{(\sec x+1)}\,dx=\cos^{-1}\frac{2\sqrt{(\sec x-1)}}{\sec x}.$

7. Integrate $\operatorname{cosec}(x-a)\operatorname{cosec}(x-b)$,

$\sin x\operatorname{cosec}(x-a)\operatorname{cosec}(x-b)$, $\operatorname{cosec}(x-a)\operatorname{cosec}(x-b)\operatorname{cosec}(x-c)$.

8. Evaluate $\displaystyle\int\frac{\operatorname{cosec}\vartheta\cot\vartheta\,d\vartheta}{\sqrt{\{\sin(\vartheta+\alpha)\sin(\vartheta-\alpha)\}}}.$

The Integrals $\displaystyle\int\frac{d\vartheta}{a+b\cos\vartheta},\ \int\frac{du}{a+b\cosh u},\ \int\frac{dv}{a+b\sinh v}.$

25. When these integrals, or more generally

$$\int\frac{d\vartheta}{a+b\cos\vartheta+c\sin\vartheta},\quad\int\frac{du}{a+b\cosh u+c\sinh u},$$

have been presented for integration, it has been customary to reduce the integration to that of a rational function of y, by means of the substitution $y=\tan\frac{1}{2}\vartheta$ or $\tanh\frac{1}{2}u$.

But following out the general idea of this paper, we shall find it more suitable to exhibit the results as functions of the function to be

integrated; thus, according as a is greater or less than b,

$$\int \frac{d\vartheta}{a+b\cos\vartheta} = \frac{1}{\sqrt{(a^2-b^2)}} \cos^{-1} \frac{a\cos\vartheta+b}{a+b\cos\vartheta},$$

$$\text{or} \quad = \frac{1}{\sqrt{(b^2-a^2)}} \cosh^{-1} \frac{a\cos\vartheta+b}{a+b\cos\vartheta};$$

$$\int \frac{d\vartheta}{a+b\sin\vartheta} = \frac{1}{\sqrt{(a^2-b^2)}} \sin^{-1} \frac{a\sin\vartheta+b}{a+b\sin\vartheta},$$

$$\text{or} \quad = \frac{1}{\sqrt{(b^2-a^2)}} \cosh^{-1} \frac{a\sin\vartheta+b}{a+b\sin\vartheta};$$

$$\int \frac{du}{a+b\cosh u} = \frac{1}{\sqrt{(a^2-b^2)}} \cosh^{-1} \frac{a\cosh u+b}{a+b\cosh u},$$

$$\text{or} \quad = \frac{1}{\sqrt{(b^2-a^2)}} \cos^{-1} \frac{a\cosh u+b}{a+b\cosh u};$$

$$\int \frac{du}{a+b\sinh u} = \frac{1}{\sqrt{(a^2+b^2)}} \sinh^{-1} \frac{a\sinh u-b}{a+b\sinh u}.$$

In the more general cases,

$$\int \frac{d\vartheta}{a+b\cos\vartheta+c\sin\vartheta}$$

$$= \frac{1}{\sqrt{(a^2-b^2-c^2)}} \cos^{-1} \frac{a\,(a+b\cos\vartheta+c\sin\vartheta)-a^2+b^2+c^2}{(a+b\cos\vartheta+c\sin\vartheta)\sqrt{(b^2+c^2)}},$$

$$\text{or} \quad = \frac{1}{\sqrt{(-a^2+b^2+c^2)}} \cosh^{-1} \frac{a\,(a+b\cos\vartheta+c\sin\vartheta)-a^2+b^2+c^2}{(a+b\cos\vartheta+c\sin\vartheta)\sqrt{(b^2+c^2)}};$$

$$\int \frac{du}{a+b\cosh u+c\sinh u}$$

$$= \frac{1}{\sqrt{(-a^2+b^2-c^2)}} \cos^{-1} \frac{a\,(a+b\cosh u+c\sinh u)-a^2+b^2-c^2}{(a+b\cosh u+c\sinh u)\sqrt{(b^2-c^2)}},$$

$$\text{or} \quad = \frac{1}{\sqrt{(a^2-b^2+c^2)}} \cosh^{-1} \frac{a\,(a+b\cosh u+c\sinh u)-a^2+b^2-c^2}{(a+b\cosh u+c\sinh u)\sqrt{(b^2-c^2)}},$$

$$\text{or} \quad = \frac{1}{\sqrt{(a^2-b^2+c^2)}} \sinh^{-1} \frac{a\,(a+b\cosh u+c\sinh u)-a^2+b^2-c^2}{(a+b\cosh u+c\sinh u)\sqrt{(c^2-b^2)}},$$

$\int \frac{d\vartheta}{a+b\cos\vartheta+c\sin\vartheta}$ being always reducible to the form

$$\int \frac{d\vartheta}{a+\sqrt{(b^2+c^2)}\cos(\vartheta+\alpha)};$$

while $$\int \frac{du}{a+b\cosh u+c\sinh u}$$
is reducible to the form
$$\int \frac{du}{a+\surd(b^2-c^2)\cosh(u+\beta)} \text{ if } b>c,$$
or $$\int \frac{du}{a+\surd(c^2-b^2)\sinh(u+\gamma)} \text{ if } b<c.$$

Generally it will be noticed that these integrals are expressed by an inverse circular, or an inverse hyperbolic function (*i.e.*, a logarithmic function), according as the denominator of the function to be integrated is not, or is capable of assuming the value zero for real values of the argument.

EXAMPLES.

1. $\int \frac{d\vartheta}{a^2\cos^2\vartheta+\beta^2\sin^2\vartheta} = \frac{1}{2a\beta}\cos^{-1}\frac{a^2\cos^2\vartheta-\beta^2\sin^2\vartheta}{a^2\cos^2\vartheta+\beta^2\sin^2\vartheta}.$

2. $\int \frac{du}{a^2\cosh^2 u+\beta^2\sinh^2 u} = \frac{1}{2a\beta}\cos^{-1}\frac{a^2\cosh^2 u-\beta^2\sinh^2 u}{a^2\cosh^2 u+\beta^2\sinh^2 u}.$

3. Determine $\int \frac{d\vartheta}{a+b\cos^2\vartheta+c\sin^2\vartheta}$ and $\int \frac{du}{a+b\cosh^2 u+c\sinh^2 u}.$

4. $$\int \frac{d\vartheta}{15\sin^2\vartheta-16\cos\vartheta} = \int \frac{d\vartheta}{(5+3\cos\vartheta)(3-5\cos\vartheta)}$$
$$= \frac{3}{34}\int \frac{d\vartheta}{5+3\cos\vartheta} + \frac{5}{34}\int \frac{d\vartheta}{3-5\cos\vartheta}$$
$$= \frac{3}{136}\cos^{-1}\frac{5\cos\vartheta+3}{5+3\cos\vartheta} + \frac{5}{136}\cosh^{-1}\frac{3\cos\vartheta-5}{3-5\cos\vartheta}.$$

5. Show how to resolve $\frac{f(\sin\vartheta,\cos\vartheta)}{\Pi(a+b\cos\vartheta+c\sin\vartheta)}$ into partial fractions, and thence to integrate it.

26. When $$\vartheta = \text{gd}\, u,$$
then, generally, $$\int (\sec\vartheta)^n\, d\vartheta = \int (\cosh u)^{n-1}\, du,$$
$$\int (\text{sech}\, u)^n\, du = \int (\cos\vartheta)^{n-1}\, d\vartheta;$$
reducing when $n=1$ to
$$\int \sec\vartheta\, d\vartheta = u = \text{gd}^{-1}\vartheta,$$
$$\int \text{sech}\, u\, du = \vartheta = \text{gd}\, u.$$

27. Similar formulæ of reduction exist for
$$\int \frac{d\vartheta}{(a+\beta\cos\vartheta)^n} \text{ and } \int \frac{du}{(a+\beta\cosh u)^n},$$
with interesting geometrical applications.

Supposing $\alpha > \beta$, then since

$$\int \frac{d\vartheta}{\alpha+\beta\cos\vartheta} = \frac{1}{\sqrt{(\alpha^2-\beta^2)}}\cos^{-1}\frac{\alpha\cos\vartheta+\beta}{\alpha+\beta\cos\vartheta},$$

if we put
$$\frac{\alpha\cos\vartheta+\beta}{\alpha+\beta\cos\vartheta} = \cos\xi,$$

we obtain the relation connecting ϑ, the *true anomaly* from perihelion, with ξ the *excentric anomaly* in an ellipse of excentricity β/α.

Then
$$\alpha+\beta\cos\vartheta = \frac{\alpha^2-\beta^2}{\alpha-\beta\cos\xi},$$

and
$$\int \frac{d\vartheta}{(\alpha+\beta\cos\vartheta)^n} = \frac{1}{(\alpha^2-\beta^2)^{n-\frac{1}{2}}}\int (\alpha-\beta\cos\xi)^{n-1}\,d\xi.$$

28. But supposing $\alpha < \beta$, then

$$\int \frac{d\vartheta}{\alpha+\beta\cos\vartheta} = \frac{1}{\sqrt{(\beta^2-\alpha^2)}}\cosh^{-1}\frac{\alpha\cos\vartheta+\beta}{\alpha+\beta\cos\vartheta};$$

so that, putting
$$\frac{\alpha\cos\vartheta+\beta}{\alpha+\beta\cos\vartheta} = \cosh\eta,$$

we obtain the relation connecting ϑ, the true anomaly from perihelion in a hyperbolic orbit of excentricity β/α, with η, which may now by analogy be called the *hyperbolic excentric anomaly*. (Prof. J. C. Adams, "Lambert's Theorem," *Messenger of Mathematics*, Vol. XII.)

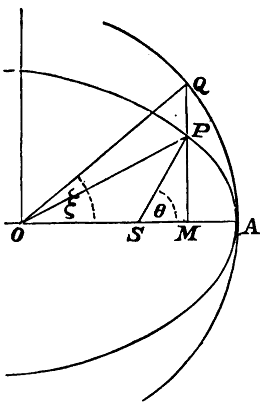

FIG. 4.

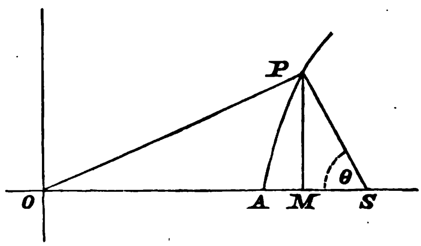

FIG. 5.

For, as in the ellipse (Fig. 4), $x^2/a^2+y^2/b^2=1$, the sectorial area $AOP=\frac{1}{2}ab\xi$, the triangle $OPS=\frac{1}{2}abc\sin\xi$, and therefore the sectorial area $ASP=\frac{1}{2}ab(\xi-e\sin\xi)$; so in the hyperbola (Fig. 5), $x^2/a^2-y^2/b^2=1$, the sectorial area $AOP=\frac{1}{2}ab\eta$, the triangle $OPS=\frac{1}{2}abe\sinh\eta$, and therefore the sectorial area $ASP=\frac{1}{2}ab(e\sinh\eta-\eta)$.

Relations between True, Excentric, and Mean Anomaly.

29. The well-known relation for the mean anomaly nt in terms of the excentric anomaly ξ in an elliptic orbit, namely

$$nt = \xi - e \sin \xi,$$

becomes for the hyperbolic orbit, by analogy,

$$nt = e \sinh \eta - \eta,$$

where

$$n^2 a^3 = \mu = h^2/l,$$

l denoting the semi-latus rectum of the orbit, elliptic or hyperbolic, and h twice the area swept out in unit time about the focus.

We notice that

$$n\frac{dt}{d\xi} = 1 - e\cos\xi = \frac{r}{a}, \quad \text{or} \quad r\frac{d\xi}{dt} = an, \text{ in the elliptic orbit,}$$

and $$n\frac{dt}{d\eta} = e\cosh\eta - 1 = \frac{r}{a}, \quad \text{or} \quad r\frac{d\eta}{dt} = an, \text{ in the hyperbolic orbit;}$$

r denoting the focal distance SP.

30. Then, when $\alpha < \beta$,

$$\alpha + \beta\cos\vartheta = \frac{\beta^2 - \alpha^2}{\beta\cosh\eta - \alpha},$$

and

$$\int \frac{d\vartheta}{(\alpha+\beta\cos\vartheta)^n} = \frac{1}{(\beta^2-\alpha^2)^{n-\frac{1}{2}}} \int (\beta\cosh\eta - \alpha)^{n-1}\, d\eta.$$

Reciprocally,

$$\int \frac{d\eta}{(\beta\cosh\eta-\alpha)^n} = \frac{1}{(\beta^2-\alpha^2)^{n-\frac{1}{2}}} \int (\alpha+\beta\cos\vartheta)^{n-1}\, d\vartheta,$$

$$\int \frac{d\eta}{(\alpha\cosh\eta-\beta)^n} = \frac{1}{(\beta^2-\alpha^2)^{n-\frac{1}{2}}} \int (\alpha+\beta\cos\vartheta)^{n-1}(\sec\vartheta)^n\, d\vartheta,$$

including the reduction of all possible cases of the integral

$$\int \frac{du}{(a+b\cosh u)^n},$$

and reducing when $\alpha = 0$, $\beta = 1$ to the cases considered in § 25.

31. The case of $n = 2$ is important in Physical Astronomy, because required in the determination of the time t or the mean anomaly nt from perihelion in terms of the true anomaly ϑ.

Then, if $e < 1$,

$$\int \frac{d\vartheta}{(1+e\cos\vartheta)^2} = \frac{1}{(1-e^2)^{\frac{3}{2}}} \int (1 - e\cos\xi)\, d\xi = \frac{1}{(1-e^2)^{\frac{3}{2}}} (\xi - e\sin\xi),$$

giving as before, in the elliptic orbit,

$$nt = \xi - e\sin\xi.$$

But
$$\cos\xi = \frac{\cos\vartheta + e}{1+e\cos\vartheta},$$
$$\sin\xi = \frac{\sqrt{(1-e^2)}\sin\vartheta}{1+e\cos\vartheta},$$
so that
$$nt = \sin^{-1}\frac{\sqrt{(1-e^2)}\sin\vartheta}{1+e\cos\vartheta} - \frac{e\sqrt{(1-e^2)}\sin\vartheta}{1+e\cos\vartheta}.$$
But
$$\int\frac{d\vartheta}{(1+e\cos\vartheta)^2} = \frac{1}{(e^2-1)^{\frac{3}{2}}}\int(e\cosh\eta - 1)\,d\eta$$
$$= \frac{1}{(e^2-1)^{\frac{3}{2}}}(e\sinh\eta - \eta),$$
if $e > 1$, giving, in the hyperbolic orbit,
$$nt = e\sinh\eta - \eta.$$
But
$$\cosh\eta = \frac{\cos\vartheta + e}{1+e\cos\vartheta},$$
$$\sinh\eta = \frac{\sqrt{(e^2-1)}\sin\vartheta}{1+e\cos\vartheta},$$
so that
$$nt = \frac{e\sqrt{(e^2-1)}\sin\vartheta}{1+e\cos\vartheta} - \sinh^{-1}\frac{\sqrt{(e^2-1)}\sin\vartheta}{1+e\cos\vartheta},$$
a form more symmetrical than that involving the logarithm of a complicated argument.

When the anomalies in an elliptic orbit are measured from aphelion,
$$nt = \xi + e\sin\xi = \sin^{-1}\frac{\sqrt{(1-e^2)}\sin\vartheta}{1-e\cos\vartheta} + \frac{e\sqrt{(1-e^2)}\sin\vartheta}{1-e\cos\vartheta};$$
and in corresponding hyperbolic orbit, which must now be supposed described under a repulsion from the focus,
$$nt = e\sinh\eta + \eta = \frac{e\sqrt{(e^2-1)}\sin\vartheta}{e\cos\vartheta - 1} + \sinh^{-1}\frac{e\sqrt{(e^2-1)}\sin\vartheta}{e\cos\vartheta - 1}.$$

When $e = 1$, the ellipse or hyperbola reduces to a straight line, and we obtain the motion of a body from rest under the attraction or repulsion of a fixed centre varying inversely as the square of the distance (Clifford, *Dynamic*, p. 111).

If $2a$ is the initial distance, and x the distance at the time t, then

(i.) if attracted, $x = a(1+\cos\xi)$, $nt = \xi + \sin\xi$,

(ii.) if repelled, $x = a(\cosh\eta + 1)$, $nt = \sinh\eta + \eta$,

and $n^2a^3 = \mu$, the strength of the centre of attraction.

The time of falling to the centre, if attracted, is obtained by putting $\xi = \pi$, and then $t = \pi/n = \pi a^{\frac{3}{2}}/\mu^{\frac{1}{2}}$.

EXAMPLES.

1. $\displaystyle\int \frac{d\vartheta}{(5+3\cos\vartheta)^2}$ (representing the time in an elliptic orbit)

$$= \frac{5}{64}\sin^{-1}\frac{4\sin\vartheta}{5+3\cos\vartheta} - \frac{3}{64}\,\frac{4\sin\vartheta}{5+3\cos\vartheta}.$$

2. $\displaystyle\int \frac{d\vartheta}{(3+5\cos\vartheta)^2}$ (representing the time in a hyperbolic orbit)

$$= \frac{5}{64}\,\frac{4\sin\vartheta}{3+5\cos\vartheta} - \frac{3}{64}\sinh^{-1}\frac{4\sin\vartheta}{3+5\cos\vartheta}.$$

3. Prove that

$$\int_0^\pi \frac{d\phi}{(\cosh x - \sinh x\cos\phi)^{n+\frac{1}{2}}} = \int_0^\pi (\cosh x - \sinh x \cos\phi)^{n-\frac{1}{2}}\,d\phi.$$

4. Prove that the area of each of the two equal and similar pieces of the ellipse $x^2/a^2+y^2/b^2=1$, which are cut off by the hyperbola $x^2/\alpha^2-y^2/\beta^2=1$ $(\alpha<a)$, is

$$ab\sin^{-1}\frac{\beta\sqrt{(a^2-\alpha^2)}}{\sqrt{(a^2\beta^2+\alpha^2b^2)}} - \alpha\beta\sinh^{-1}\frac{b\sqrt{(a^2-\alpha^2)}}{\sqrt{(a^2\beta^2+\alpha^2b^2)}}.$$

5. Evaluate $\displaystyle\int \frac{d\vartheta}{(a+b\cos\vartheta)^3}$.

6. Given with C. G. S. units that the quadrant of the Earth is 10^9, the mean density 5·67, and the acceleration of gravity is 981, prove that two homogeneous spheres, each of mass 10^6 and diameter 50, will under their gravitation take about 2525 seconds to come into contact, their centres being initially a metre apart.

The Integrals $\displaystyle\int \frac{d\vartheta}{(a+b\cos\vartheta+c\sin\vartheta)^n}$, $\displaystyle\int \frac{du}{(a+b\cosh u+c\sinh u)^n}$.

32. The first integral is reducible to a preceding form

$$\int \frac{d\vartheta}{\{a+e\cos(\vartheta-\gamma)\}^n},$$

by putting $b=e\cos\gamma$, $c=e\sin\gamma$; and can thus be expressed as a function of $a+b\cos\vartheta+c\sin\vartheta$.

But the integral

$$\int \frac{du}{(a+b\cosh u+c\sinh u)^n}$$

is reducible to

$$\int \frac{du}{\{a+e\cosh(u+\delta)\}^n},$$

by putting $b=e\cosh\delta$, $c=e\sinh\delta$, only when $b>c$; the result being expressed as a function of $a+b\cosh u+c\sinh u$.

When $b<c$, then this integral is reducible to

$$\int \frac{du}{\{a+e\sinh(u+\delta)\}^n},$$

by putting $b = e \sinh u$, $c = e \cosh u$; and therefore depends on

The Integral $\int \frac{du}{(\alpha+\beta \sinh u)^n}$.

33. Now $\int \frac{du}{\alpha+\beta \sinh u} = \frac{1}{\sqrt{(\alpha^2+\beta^2)}} \sinh^{-1} \frac{\alpha \sinh u - \beta}{\alpha+\beta \sinh u}$;

so that, putting $$\frac{\alpha \sinh u - \beta}{\alpha + \beta \sinh u} = \sinh v,$$

then $$\sinh u = \frac{\alpha \sinh v + \beta}{\alpha - \beta \sinh v}, \quad \alpha + \beta \sinh u = \frac{\alpha^2+\beta^2}{\alpha - \beta \sinh v},$$

and $$\int \frac{du}{(\alpha+\beta \sinh u)^n} = \frac{1}{(\alpha^2+\beta^2)^{n+\frac{1}{2}}} \int (\alpha - \beta \sinh v)^{n-1} \, dv.$$

To interpret geometrically this relation connecting u and v, let $\vartheta = \operatorname{gd} u$, $\phi = \operatorname{gd} v$, so that $\sinh u = \tan \vartheta$, $\sinh v = \tan \phi$; then

$$\tan \vartheta = \frac{\alpha \tan \phi + \beta}{\alpha - \beta \tan \phi} = \tan(\phi + \gamma),$$

putting $$\beta / \alpha = \tan \gamma,$$

so that $$\vartheta = \phi + \gamma.$$

Then $$\int \frac{du}{(\alpha+\beta \sinh u)^n} = \frac{1}{(\alpha^2+\beta^2)^{\frac{1}{2}n}} \int \frac{\sec \vartheta \, d\vartheta}{(\cos \gamma + \sin \gamma \tan \vartheta)^n}$$
$$= \frac{1}{(\alpha^2+\beta^2)^{\frac{1}{2}n}} \int (\cos \vartheta)^{n-1} (\sec \phi)^n \, d\vartheta \text{ (or } d\phi).$$

Abel's Theorem and the Rectification of Curves.

34. It follows from Abel's theorem (Bertrand, *Calcul Intégral*, p. 106) that if $\int \frac{N}{D} \frac{dx}{\sqrt{R}}$ is integrable, where D, N, R are rational integral functions of x, then

$$\int \frac{N}{D} \frac{dx}{\sqrt{R}} = \eta + \vartheta \sqrt{R} + A \log(\alpha + \beta \sqrt{R}) + B \log(\gamma + \delta \sqrt{R}) + \ldots,$$

where η, ϑ, α, β, γ, δ, ... are rational functions of x.

Changing the sign of $\sqrt{R}$, then

$$-\int \frac{N}{D} \frac{dx}{\sqrt{R}} = \eta - \vartheta \sqrt{R} + A \log(\alpha - \beta \sqrt{R}) + B \log(\gamma - \delta \sqrt{R}) + \ldots,$$

so that, subtracting,

$$\int \frac{N}{D} \frac{dx}{\sqrt{R}} = \vartheta \sqrt{R} + \tfrac{1}{2} A \log \frac{\alpha + \beta \sqrt{R}}{\alpha - \beta \sqrt{R}} + \tfrac{1}{2} B \log \frac{\gamma + \delta \sqrt{R}}{\gamma - \delta \sqrt{R}} + \ldots,$$

or, with the notation of the inverse hyperbolic functions,

$$= \vartheta \sqrt{R} - A \tanh^{-1} \frac{\beta}{\alpha} \sqrt{R} - B \tanh^{-1} \frac{\delta}{\gamma} \sqrt{R} \ldots;$$

and the inverse hyperbolic functions $\tanh^{-1}$ must be replaced by inverse circular functions $\tan^{-1}$, if necessary, to throw the integral into a real form.

The geometrical interpretation of this general integral may be taken to be the rectification of a unicursal curve, whose coordinates are given by

$$x : y : 1 = A : B : C,$$

where A, B, C represent rational integral functions of a variable t.

Then $$\frac{dx}{dt} = \frac{A'C - AC'}{C^2}, \quad \frac{dy}{dt} = \frac{B'C - BC'}{C^2},$$

and $$\frac{ds}{dt} = \frac{\sqrt{\{(A'C - AC')^2 + (B'C - BC')^2\}}}{C^2},$$

and the curve is rectifiable when this expression is integrable (*vide* "Rectification of Certain Curves," by R. A. Roberts, *Proceedings of the London Mathematical Society*, November, 1886).

Abel (*Œuvres*, p. 104) has considered the particular case when this *pseudo-elliptic* integral reduces to a single term, and then

$$\int \frac{M}{N} \frac{dx}{\sqrt{R}} = A \tanh^{-1} \frac{q}{p} \sqrt{R} \text{ or } A \tan^{-1} \frac{q}{p} \sqrt{R}.$$

EXAMPLES.

1. Prove that, if $$R = ax^4 + 2bx^3 + cx^2 + 2bx + a,$$
a recurring biquadratic

$$\int \frac{x^2-1}{x} \frac{dx}{\sqrt{R}} = \frac{1}{\sqrt{a}} \cosh^{-1} \frac{\sqrt{a}\sqrt{R}}{x\sqrt{\Delta}},$$

$$\text{or } = \frac{1}{\sqrt{a}} \sinh^{-1} \frac{\sqrt{a}\sqrt{R}}{x\sqrt{(-\Delta)}},$$

$$\text{or } = \frac{1}{\sqrt{(-a)}} \sin^{-1} \frac{\sqrt{(-a)}\sqrt{R}}{x\sqrt{(-\Delta)}};$$

where $\Delta = ac - b^2 - 2a^2$,

so that $aR = (ax^2 + bx + a)^2 + \Delta x^2$.

2. $$\int \frac{x^2-1}{x^2+1} \frac{dx}{\sqrt{R}} = \frac{1}{\sqrt{(c-2a)}} \cosh^{-1} \frac{\sqrt{(c-2a)}\sqrt{R}}{(x^2+1)\sqrt{\Delta}},$$

$$\text{or } = \frac{1}{\sqrt{(c-2a)}} \sinh^{-1} \frac{\sqrt{(c-2a)}\sqrt{R}}{(x^2+1)\sqrt{(-\Delta)}},$$

$$\text{or } = \frac{1}{\sqrt{(2a-c)}} \sin^{-1} \frac{\sqrt{(2a-c)}\sqrt{R}}{(x^2+1)\sqrt{(-\Delta)}};$$

and $(c-2a)R = \{bx^2 + (c-2a)x + b\}^2 + \Delta(x^2+1)^2$.

3. Determine in a similar manner

$$\int \frac{x^2+1}{x} \frac{dx}{\sqrt{R'}} \text{ and } \int \frac{x^2+1}{x^2-1} \frac{dx}{\sqrt{R'}},$$

where $R' = ax^4 - 2bx^3 + cx^2 + bx + a$.

4. Prove that (Clausen)

$$\int \frac{x\,dx}{(x^3+8)\sqrt{(x^3-1)}} = \frac{1}{18}\tan^{-1}\frac{3x(x-1)}{(x-4)\sqrt{(x^3-1)}} - \frac{1}{18}\sqrt{3}\tanh^{-1}\frac{\sqrt{3}(x-1)}{\sqrt{(x^3-1)}}.$$

5. $$\int \frac{x\,dx}{\sqrt{(x^4+2x^3-3x^2-ax+a)}} = \tfrac{1}{2}\tanh^{-1}\frac{(x+2)\sqrt{R}}{x^3+3x^2-2-\tfrac{1}{2}a}$$

$$= \tfrac{1}{2}\sinh^{-1}\frac{\sqrt{(x^3+3x^2-a)}}{\sqrt{(a-4)}} \text{ or } = \tfrac{1}{2}\sinh^{-1}\frac{(x+2)\sqrt{(x-1)}}{\sqrt{(4-a)}} \text{ (Abel).}$$

6. $$\int \frac{(x+q)\,dx}{\sqrt{\{(x^2+2qx+p)(x^2+2qx+p')\}}} = \cosh^{-1}\frac{\sqrt{(x^2+2qx+p)}}{\sqrt{(p-p')}}$$

$$\text{or } = \sinh^{-1}\frac{\sqrt{(x^2+2qx+p)}}{\sqrt{(p'-p)}} \text{ (Abel).}$$

7. $$\int \frac{x\,dx}{\sqrt{\{(x^2+2qx-q^2-2qq')(x^2+2q'x-q'^2-2qq')\}}}$$

$$= \tfrac{1}{2}\tanh^{-1}\frac{(x+2q+q')\sqrt{(x^2+2q'x-q'^2-2qq')}}{(x+q+2q')\sqrt{(x^2+2qx-q^2-2qq')}} \text{ (Abel).}$$

8. Rectify the inverse of a parabola with respect to any point; or the inverse of an epi- or hypo-cycloid with respect to the centre (R. A. Roberts, *Proc. Lond. Math. Soc.*, 1886, p. 102).

35. The hyperbolic functions may be used in conjunction with the circular functions in order to give a more symmetrical form to the result of the integration of $\int \frac{x^{m-1}dx}{1 \pm x^{2n}}$.

Even values of m may be excluded, as the integral is then reducible to a function of x^2; supposing then that m is odd,

$$\int_0 \frac{x^{m-1}dx}{1+x^{2n}} = \frac{1}{2n}\sum_{r=1}^{r=n}\cos\frac{2r-1}{2n}m\pi\tanh^{-1}\left(\frac{2x\cos\frac{2r-1}{2n}\pi}{1+x^2}\right)$$

$$+\frac{1}{2n}\Sigma\sin\frac{2r-1}{2n}m\pi\tan^{-1}\left(\frac{2x\sin\frac{2r-1}{2n}\pi}{1-x^2}\right);$$

$$\int_0 \frac{x^{m-1}dx}{1-x^{2n}} = -\frac{1}{n}\tanh^{-1}x - \frac{1}{2n}\sum_{r=1}^{r=n-1}\cos\frac{rm\pi}{n}\tanh^{-1}\frac{2x\cos\frac{r\pi}{n}}{1+x^2}$$

$$+\frac{1}{2n}\sum_{r=1}^{r=n-1}\sin\frac{rm\pi}{n}\tan^{-1}\frac{2x\sin\frac{r\pi}{n}}{1-x^2},$$

or $= -\frac{1}{m}\coth^{-1}x - \ldots$ according as $x^2 \lessgtr 1$.

The results of the integration of $\int \frac{x^{m-1}dx}{1 \pm x^{2n+1}}$ cannot be combined so as to be exhibited in this manner.